RELAXATION SPECTRUM RECOVERY USING FOURIER TRANSFORMS

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Declarations.

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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Abstract

In this thesis we consider the problem of recovering the relaxation spectrum from the storage and loss moduli. We invert an integral equation using Fourier transforms. Recovering the relaxation spectrum is an inverse, ill-posed problem and hence regularisation methods must be used to try and obtain the relaxation spectrum. We are particularily interested in establishing properties of the relaxation spectrum. We note from the literature that there are results of compact support for the relaxation spectrum; we review to what extent and in what sense, these results are valid. We consider the methods used in the literature and demonstrate their strengths and weaknesses, supplying some missing details.

We demonstrate in chapter 3 the difficulty in obtaining an interval of compact support for the relaxation spectrum and in the remainder of chapter 3 and chapter 4 we prove results of non-compactness of support for non-trivial relaxation spectra. Our settings are square integrable functions in chapter 3, and Schwartz distributions in chapter 4; we make use of Paley-Wiener theorems. These are important results since they contradict results in the literature that we review in chapter 2. We are able to demonstrate, using examples and via direct calculations, that the relaxation spectrum becomes insignificant outside some closed interval. With regards to numerical computations, this could be considered as a weak form of compact support. We call this essential numerical support; this may be a useful concept for the practical rheologist.

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Chapter 1

Introduction

1.1 The Problem

In this thesis we will be considering the following problem:

$$g = \operatorname{sech} * h \tag{1.1}$$

where we would like to find h or properties of h, given some g, where $*$ denotes convolution.

We take Fourier transforms of both sides and, assuming that we are working in reasonable spaces, then we can express the Fourier transform of a convolution as the product of two Fourier transforms, for detail see Rudin [71].

$$\hat{g} = \widehat{\operatorname{sech} * h} = \widehat{\operatorname{sech}} \cdot \hat{h} \tag{1.2}$$

The sech function is almost a self Fourier transform and we write $\xi_\lambda(r) = \cosh(\lambda r)$. We note that λ is the constant $\pi/2$ for our problem. We can express equation (1.2) as:

$$\hat{g} \cdot \xi_\lambda = \hat{h}. \tag{1.3}$$

Finally, we take inverse Fourier transforms of both sides to give:

$$\mathscr{F}^{-1}\left[\hat{g} \cdot \xi_\lambda\right] = h. \tag{1.4}$$

The main motivation for this research are the results of Loy, Newbury, Davies and Anderssen [50],[51], Dodd [23] and Renardy [69]. The main methods used by Loy, Newbury, Davies and Anderssen [50],[51] and Dodd [23] in recovering h from the above equation, are the methods that we will also be considering in this thesis.

The aim of this thesis is to make sense of the work done on recovering h (the continuous relaxation spectrum) and to evaluate the validity of these results. We will demonstrate that there are in fact some flaws in the results and we will suggest an alternative approach for deducing information about the support of h (the relaxation spectrum).

1.2 Inverse and Ill-posed Problems

The problem we introduced in equation (1.1) can be generalised to a problem of the following form:

$$\int_a^b k(x,\tau) f(\tau)\ \mathrm{d}\tau = g(x) \tag{1.5}$$

for $a < x < b$. This is known as a Fredholm integral equation of the first kind, where $g = g(x)$ is the *data function*, $k(x,\tau)$ is the *kernel function* and f is the function we are trying to find.

Some important points to note are that, firstly, the existence of solutions is not obvious. The kernel k acts as a smoothing operator and this smoothness is attained by the function g. Hence, if the function g is not continuous, but the kernel is, then equation (1.5) can have no integrable solution. This is a cause of ill-posedness.

Furthermore, if a solution does exist, it may not be a unique solution, e.g. if the kernel is $k(x,\tau) = x \sin \tau$ then we will have an infinite number of solutions for f. Finally, solutions of equation (1.5), in general, depend discontinuously on the data. We will mention this in more detail later on. For further detail on Fredholm integral equations see Groetsch [36].

Equation (1.5) is known as an inverse, ill-posed problem. We think of $g(x)$ as the output in equation (1.5) and $f(\tau)$ as the input. The forward problem would be to calculate the function g given f and k. However, we are interested in solving equation (1.5) for f, i.e., we want to calculate f given some g and k. This is known as the inverse problem. The concept of ill-posed was first introduced by Hadamard back in 1923. He defined a well posed problem as a problem that:

- has a solution (existence);

- the solution is unique (uniqueness); and
- the solution depends continuously on the data, in some reasonable topology (stability of solution).

Hadamard defines an ill-posed problem as a problem that fails on any one or more of the above criteria. As we will see later, our problem fails on the last point, where small perturbations in the data functional g may have a big affect on the solution f (this means that the solution is not robust against noise). Regularisation methods can be utilized to overcome this. We will describe this in more detail later.

Additional detail on the theory and application of ill-posed problems can be found in Bakushinsky and Goncharsky [7], Murio [62] and Tikhonov and Arsenin [78].

1.3 Viscoelastic Materials

The problem we are considering in this thesis is a well known problem that arises in the study of viscoelastic fluids. Viscoelastic fluids are materials that display both solid-like and liquid-like properties. They are widely used in industry, e.g. viscoelastic fluids are used in processes such as extrusion and injection molding of plastics, coating and lubricating. The behavior of such liquids can be characterized as 'wobbly' and 'stringy' and such fluids display a rod climbing effect on stirring. For more detail see Barnes [8] and [9]. Paint, for example, appears solid after storage, however, upon stirring or brushing behaves like a fluid, and finally, it re-thickens and sets after painting. Ink, cement, shampoo, cleaning products and toothpaste are all examples of viscoelastic fluids. Other examples of viscoelastic materials occur widely in the food industry, for example, honey, salad cream, ketchup, cream etc. It is clear then that being able to model such materials would be of great interest to many. In fact knowledge of parameters involved in the processes dealing with viscoelastic fluids can improve the quality of products.

1.3.1 Models of Viscoelastic Fluids

Since viscoelastic fluids play such an important role in industry, understanding their properties and how they behave is of great importance. Hence, over the years many fluid models have been suggested in order to predict properties of these materials. When we apply a stress or strain to a material, we expect to see a certain response depending on the type of material we are dealing with. If we are dealing with a purely elastic material, i.e. materials that satisfy Hooke's law, then we would observe an extension proportional to the force applied to the material. Furthermore, the energy used in the deformation is stored in the material and if the force is removed, this energy can be recovered. If we had a purely viscous material (Newtonian fluid) and applied a stress or strain we would see an almost instantaneous deformation (flow). The energy used to produce the deformation is lost as heat (for more detail see Tschoegl [81] and Bland [14]). These two extreme cases are often modelled by a Hookean spring, for purely elastic materials and a dashpot for purely viscous materials. In principle, all real materials fall between these two extreme cases i.e. are viscoelastic, since, under certain circumstances, some energy may be stored during deformation.

The model we use to represent our viscoelastic fluid is the Maxwell model, which can be thought of as a dashpot (representing the Newtonian element) in line with a Hookean spring (elastic element of fluid). For N parallel dashpot and spring elements the Maxwell model is represented by the equation (see Ferry [27] for more detail):

$$G\left(t-t^{'}\right)=\sum_{i=1}^{N}g_i\exp\left[\frac{-\left(t-t^{'}\right)}{\tau_i}\right] \tag{1.6}$$

where $G\left(t\right)$ is the linear relaxation modulus and g_i is the corresponding elastic modulus of a relaxation time τ_i, $(1\leq i\leq N)$.

$G\left(t\right)$ is a function that is directly measurable by a sudden shear displacement ($G\left(t\right)$ can be thought of as a time-dependent ratio of stress to strain), by relating the stress tensor, $\sigma\left(t\right)$ to the rate of deformation tensor, $\dot{\gamma}\left(t^{'}\right)$ in the constitutive equation:

$$\sigma\left(t\right)=\int_{-\infty}^{t}G\left(t-t^{'}\right)\dot{\gamma}\left(t^{'}\right)\,\mathrm{d}t^{'}. \tag{1.7}$$

We can see from equation (1.6) that $G\left(t\right)$ is a decreasing function (for positive τ_i). This corresponds to what we refer to as "fading memory", i.e. more recent strains would be more significant than strains from a while ago.

The type of fluids we will be considering in this work are those that are modelled by linear viscoelastic theory; a model that assumes a linear relationship between the current stress of a fluid and its strain history (for detail see Wilson[83]), which is the result of a sufficiently small strain. For more detail on linear viscoelasticity see Bland [14].

1.4 The Relaxation Spectrum

The relaxation spectrum $H(\tau)$ describes for how long and to what extent a fluid remembers its past deformation history. The relaxation spectrum is important in characterizing viscoelastic fluids. It can be used to predict the behavior of such fluids in many standard experiments (see Owens and Phillips [65]). We note that the relaxation spectrum H is related to the linear relaxation modulus G by equation (1.16).

The problem of recovering the Relaxation spectrum is that it is not an experimentally measurable quantity. It is related to two functions $G^{'}(\omega)$ and $G^{''}(\omega)$, the storage modulus and loss modulus respectively, by two Fredholm integral equations:

$$G^{'}(\omega) = \int_0^\infty \frac{\omega^2\tau^2}{1+\omega^2\tau^2}\frac{H(\tau)}{\tau}\,\mathrm{d}\tau, \tag{1.8}$$

$$G^{''}(\omega) = \int_0^\infty \frac{\omega\tau}{1+\omega^2\tau^2}\frac{H(\tau)}{\tau}\,\mathrm{d}\tau. \tag{1.9}$$

Both of these quantities can be measured from oscillatory shear experiments. Recovering the relaxation spectrum from either of these equations is an ill-posed, inverse problem. As was mentioned in section 1.2, the problem of inverting either of equations (1.8) and (1.9) is ill-posed in the sense of Hardamard in that small perturbations in the storage or loss moduli may lead to a large perturbation of the relaxation spectrum.

The relaxation spectrum cannot be measured directly, but we can however measure experimentally the storage and loss moduli. We are dealing with fluids which are well described by linear viscoelasticity for small strains. This is why the experiments performed to obtain the storage and loss modulus (for more detail see Barnes [9]) involve oscillatory shear, which can give small strains if the amplitude is kept small. Essentially, the experiments involve a pair of concentric cylinders, one is fixed and the other free to oscillate, with the fluid placed between the two cylinders. The rotation is that of simple harmonic motion which has shear displacement:

$$\gamma\left(t\right) = \alpha \sin\left(\omega t\right),$$

where γ is the shear, ω the angular frequency and α the amplitude (see Wilson [83]). We note for the remainder of the thesis we will simply refer to ω, the angular frequency, as frequency. As described above, α must be kept small to ensure we are within the linear regime. However if we are dealing with an ideal linear viscoelastic material then α can be any value. It follows that:

$$\dot{\gamma}\left(t\right) = \alpha\omega \cos\left(\omega t\right).$$

If we were to assume that this motion started a long time ago, i.e. as $t \rightarrow -\infty$, then from equation (1.7) we can write:

$$\sigma\left(t\right) = \int_{-\infty}^{t} G\left(t-t^{'}\right) \dot{\gamma}\left(t^{'}\right) \,\mathrm{d}t^{'} = \int_{-\infty}^{t} G\left(t-t^{'}\right) \alpha\omega \cos\left(\omega t^{'}\right) \,\mathrm{d}t^{'}. \tag{1.10}$$

Furthermore, let $s = t - t^{'}$ then equation (1.10) becomes:

$$\sigma(t) = -\alpha\omega \int_{\infty}^{0} G(s)\cos(\omega(t-s))\ \mathrm{d}s = \alpha\omega \int_{0}^{\infty} G(s)\cos(\omega(t-s))\ \mathrm{d}s. \qquad (1.11)$$

Next we write $\cos(\omega(t-s))$ as $\Re[\exp[i\omega(t-s)]]$, (for more detail see Owens and Phillips [65] and Wilson [83]), to give

$$\begin{aligned} \sigma(t) &= \alpha\omega \int_{0}^{\infty} G(s)\,\Re[\exp[i\omega(t-s)]]\ \mathrm{d}s = \alpha\omega\Re\left[\int_{0}^{\infty} G(s)\exp[i\omega(t-s)]\ \mathrm{d}s\right] \\ &= \alpha\omega\Re\left[\exp[i\omega t]\int_{0}^{\infty} G(s)\exp[-i\omega s]\ \mathrm{d}s\right]. \qquad (1.12) \end{aligned}$$

The integral in equation (1.12) is a one-sided Fourier transform. The integral will be a complex function of ω. Furthermore, the complex shear modulus, G^{*} is defined to be:

$$G^{*}(\omega) = i\omega \int_{0}^{\infty} G(s)\exp[-i\omega s]\ \mathrm{d}s, \qquad (1.13)$$

which has real and imaginary parts: $G^{*}(\omega) = G^{'}(\omega) + iG^{''}(\omega)$, where $G^{'}(\omega)$ is the storage modulus and $G^{''}(\omega)$ the loss modulus. It follows from equation (1.13) that these can be expressed as:

$$G^{'}(\omega) = \omega \int_{0}^{\infty} G(s)\sin(\omega s)\ \mathrm{d}s, \qquad (1.14)$$

$$G^{''}(\omega) = \omega \int_0^{\infty} G(s) \cos(\omega s) \, \mathrm{d}s. \tag{1.15}$$

For a variety of reasons it is often useful to express the linear relaxation function $G(s)$ in terms of a distribution function (spectrum), $H(\tau)$ of relaxation times τ. The Relaxation Spectrum $H(\tau)$ may be continuous or discrete. The relaxation modulus, $G(s)$ is defined in terms of the relaxation spectrum $H(\tau)$ by the following expression:

$$G(s) = \int_0^{\infty} \frac{H(\tau)}{\tau} \exp\left(\frac{-s}{\tau}\right) \, \mathrm{d}\tau. \tag{1.16}$$

If we substitute equation (1.16) into equations (1.14) and (1.15) we obtain:

$$G^{'}(\omega) = \omega \int_0^{\infty} \int_0^{\infty} \frac{H(\tau)}{\tau} \exp\left(\frac{-s}{\tau}\right) \, \mathrm{d}\tau \sin(\omega s) \, \mathrm{d}s, \tag{1.17}$$

$$G^{''}(\omega) = \omega \int_0^{\infty} \int_0^{\infty} \frac{H(\tau)}{\tau} \exp\left(\frac{-s}{\tau}\right) \, \mathrm{d}\tau \cos(\omega s) \, \mathrm{d}s. \tag{1.18}$$

We consider equation (1.17) and noting that we can use Fubini's Theorem (Theorem 2.4 in chapter 2) to interchange the order of integration (from the fact that the integral is finite with respect to s and τ $(=G^{'}(\omega))$) we can write:

$$G^{'}(\omega) = \omega \int_0^\infty \frac{H(\tau)}{\tau} \int_0^\infty \exp\left(\frac{-s}{\tau}\right) \sin(\omega s)\, \mathrm{d}s\, \mathrm{d}\tau. \tag{1.19}$$

We consider the inner integral:

$$\int_0^\infty \exp\left(\frac{-s}{\tau}\right) \sin(\omega s)\, \mathrm{d}s.$$

We make use of integration by parts, where:

$$u = \sin(\omega s) \text{ and hence } \frac{\mathrm{d}u}{\mathrm{d}s} = \omega \cos(\omega s)$$

and

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \exp\left\{\frac{-s}{\tau}\right\} \text{ and hence } v = -\tau \exp\left\{\frac{-s}{\tau}\right\}.$$

It follows that:

$$I_1 = \int_0^\infty e^{-s/\tau} \sin(\omega s)\, \mathrm{d}s = \underbrace{\left[-\tau e^{-s/\tau} \sin(\omega s)\right]_0^\infty}_{=0} + \tau\omega \int_0^\infty e^{-s/\tau} \cos(\omega s)\, \mathrm{d}s. \tag{1.20}$$

We use the method of integration by parts to evaluate:

$$\int_0^\infty e^{-s/\tau} \cos(\omega s) \, \mathrm{d}s$$

where

$$u = \cos(\omega s) \text{ and hence } \frac{\mathrm{d}u}{\mathrm{d}s} = -\omega \sin(\omega s)$$

and

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \exp\left\{\frac{-s}{\tau}\right\} \text{ and hence } v = -\tau \exp\left\{\frac{-s}{\tau}\right\}.$$

It follows that:

$$\begin{aligned} I_1 &= \int_0^\infty e^{-s/\tau} \sin(\omega s) \, \mathrm{d}s = \tau\omega \int_0^\infty e^{-s/\tau} \cos(\omega s) \, \mathrm{d}s \\ &= \omega\tau \left[-\tau e^{-s/\tau} \cos(\omega s)\right]_0^\infty - \omega\tau \int_0^\infty \omega \sin(\omega s) \, \tau e^{-s/\tau} \, \mathrm{d}s \\ &= \omega\tau^2 - \omega^2\tau^2 I_1. \end{aligned} \tag{1.21}$$

We have that

$$I_1 \left[1 + \omega^2\tau^2\right] = \omega\tau^2,$$

$$I_1 = \frac{\omega\tau^2}{1+\omega^2\tau^2}.$$

If we substitute this back into equation (1.17), we obtain:

$$G^{'}(\omega) = \omega\int_0^\infty \frac{H(\tau)}{\tau}\frac{\omega\tau^2}{1+\omega^2\tau^2}\,\mathrm{d}\tau = \int_0^\infty \frac{H(\tau)}{\tau}\frac{\omega^2\tau^2}{1+\omega^2\tau^2}\,\mathrm{d}\tau. \tag{1.22}$$

Thus we have derived equation (1.8); analogous calculations yield equation (1.9).

For more detail on the background of viscoelastic fluids see Ferry [27], Owens and Phillips [65], Schowalter [74], Tschoegl [81] and Bland [14].

1.5 Recovering the Relaxation Spectrum from Storage and Loss Moduli

We noted in previous sections that the problem of recovering the relaxation spectrum from either of equations (1.8) and/or (1.9) is ill-posed. It is ill-posed in the sense that small perturbations in either $G^{'}(\omega)$ or $G^{''}(\omega)$ may lead to a large perturbation of $H(\tau)$. As a result, recovery methods need to involve some form of regularisation. This research is involved with recovering the continuous relaxation spectrum and the majority of literature studied has been with regards to the continuous relaxation spectrum. However, there is much literature involving the recovery of the relaxation spectrum via discrete methods. Should the reader wish to know more about these discrete methods they may refer to the work of Honerkamp and Weese [41], Baumgaertel and Winter [11], Hussein

[43], Haghtalab and Sodeifian [37], Davies and Anderssen [5],[21] and Newbury [63].

For the remainder of this chapter we will be mentioning the work that has been done in this field in recovering the continuous relaxation spectrum from the storage and loss moduli. In the following chapter we will be going into further detail on the methods used to recover and deduce properties of the continuous relaxation spectrum.

We could have used either or both of equations (1.8) and (1.9) to recover the relaxation spectrum. We use equation (1.9) only, from which we derive equation (1.1), to recover the relaxation spectrum. In fact, the majority of literature that recovers the continuous relaxation spectrum and/or its properties, favours the loss modulus over the storage modulus. Many authors attempt inverting both equations, but quite often it is claimed that the loss modulus is the more accurate/easier option. Hussein [43] shows that through using a convolution filter method, it is easier to estimate the relaxation spectrum from the loss modulus than it is from the storage modulus. Haghtalab and Sodeifian [37] conclude that their method has higher accuracy for the loss modulus and Al-Aidarous [4] attempts to invert both integral equations and demonstrates that the loss modulus is much easier to deal with. Loy, Newbury, Anderssen and Davies [50] consider only the loss modulus to get an expression for the relaxation spectrum as does Dodd [23]. Davies and Anderssen note that in Honerkamp and Weese's double Gaussian spectrum, the noise level on the loss moduli is considerably less than on the storage moduli, at high frequencies. Newbury [63] also considers the inversion of the loss modulus.

We demonstrate calculations for both and explain why it is the loss modulus that we will consider for the remainder of this work. The calculations involving the loss modulus are seen in part in the work of Loy, Newbury, Davies and Anderssen [50],[51] and Dodd [23].

1.5.1 Recovering h via the Loss Modulus

We refer back to equation (1.8).

$$G''(\omega) = \int_0^\infty \frac{\omega\tau}{1+\omega^2\tau^2}\frac{H(\tau)}{\tau}\,\mathrm{d}\tau. \tag{1.23}$$

We make the following substitutions:

Define; $\tau := e^\mu$ and $\omega := e^{-\nu}$, allowing us to write; $h(\mu) := H(e^\mu)$ and $g_2(\nu) := G''(e^{-\nu})$.

We substitute these into our equation above and after some manipulation, we obtain;

$$g_2(\nu) = \int_{-\infty}^\infty \frac{1}{2}\mathrm{sech}(\nu-\mu)h(\mu)\,\mathrm{d}\mu. \tag{1.24}$$

If we define the function $k_2(\nu-\mu) := \frac{1}{2}\mathrm{sech}(\nu-\mu)$ then our equation can be written as;

$$g_2(\nu) = \int_{-\infty}^\infty k_2(\nu-\mu)h(\mu)\,\mathrm{d}\mu \equiv k_2 * h. \tag{1.25}$$

Note that this is the definition of the convolution of k_2 with h.

Furthermore, we know from various properties of Fourier transforms and convolutions, omitting any regularity considerations, that we may write $\widehat{k_2 * h} = \hat{k_2}\hat{h}$, where $\hat{k_2}$ is the Fourier transform of k_2 and $k_2 * h$ is the convolution of k_2 with h.

So what we now have is that;

$$g_2(\nu) = (k_2 * h)(\nu) \tag{1.26}$$

and by taking Fourier transforms of both sides we get;

$$\hat{g}_2 = \widehat{k_2 * h} = \hat{k}_2 \hat{h}. \tag{1.27}$$

It would then seem that by taking inverse Fourier transforms, we could obtain the function that we require, that is $h = \mathscr{F}^{-1}\left[\left(\hat{k}_2\right)^{-1}\hat{g}_2\right]$ where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform.

When can this formal calculation be made rigorous? In particular, does the inverse transform $h = \mathscr{F}^{-1}\left[\left(\hat{k}_2\right)^{-1}\hat{g}_2\right]$ make sense?

The method of inversion that we describe is via Fourier transforms; when we inverse transform we wish to work in certain function spaces. This naturally imposes growth conditions as $|\nu| \to \infty$.

Now, our function $k_2(\cdot)$ is defined to be $\mathrm{sech}(\cdot)$, and we can calculate the Fourier transform of this to be;

$$\mathscr{F}\left[\mathrm{sech}\right](p) = \pi \mathrm{sech}\, \frac{p\pi}{2}. \tag{1.28}$$

See Fourier Transform 1 in the Appendix for calculations.

We now have;

$$\hat{h}\left(p\right) = \frac{2}{\pi}\cosh(\frac{\pi p}{2})\hat{g}_2\left(p\right). \tag{1.29}$$

Finally, provided that the inverse Fourier transform exists, we can obtain h to be;

$$h = \mathscr{F}^{-1}\left[\frac{2}{\pi}\cosh(\frac{\pi p}{2})\hat{g}_2\right]. \tag{1.30}$$

Cosh is a rapidly increasing function as $|p| \to \infty$, which imposes decay conditions on $\hat{g}_2$ if we wish the product to be in a reasonable space.

1.5.2 Recovering h via the Storage Modulus

Now we perform similar calculations for equation (1.9).

We consider the following equation;

$$G^{'}\left(\omega\right) = \int_0^\infty \frac{\omega^2\tau^2}{1+\omega^2\tau^2}\frac{H(\tau)}{\tau}\,\mathrm{d}\tau. \tag{1.31}$$

We make a change of variables. Define $\tau = e^\mu$ and $\omega = e^{-\chi}$.

Furthermore, let $h\left(\mu\right) := H\left(e^\mu\right)$ and $g_1\left(\chi\right) := G^{'}\left(e^{-\chi}\right)$, then after some rearranging, our Fredholm equation becomes;

$$g_1(\chi) = \int_{-\infty}^\infty \frac{1}{1+e^{2(\chi-\mu)}}h(\mu)\,\mathrm{d}\mu. \tag{1.32}$$

Now

$$\frac{1}{2}\left(1+\tanh\left(x\right)\right)=\frac{1}{1+e^{-2x}}.$$

Hence, we can write equation (1.32) as:

$$g_1(\chi)=\frac{1}{2}\int_{-\infty}^{\infty}\left[1+\tanh\left(\mu-\chi\right)\right]h(\mu)\,\mathrm{d}\mu. \tag{1.33}$$

The kernel of this Fredholm integral equation is:

$$k_1\left(\mu,\chi\right):=\left(1+\tanh\left(\mu-\chi\right)\right).$$

We write equation (1.33) as;

$$g_1(\chi)=\frac{1}{2}\int_{-\infty}^{\infty}k_1\left(\mu-\chi\right)h(\mu)\,\mathrm{d}\mu. \tag{1.34}$$

This is also the definition of the convolution of k_1 with h, written $k_1 * h$, hence;

$$g_1=k_1 * h. \tag{1.35}$$

If we now take Fourier transforms of both sides, and noting that the Fourier transform of a convolution is the product of the two separate Fourier transforms [71], we can write:

$$\hat{g}_1 = \widehat{k_1 * h} = \hat{k_1}\hat{h}. \tag{1.36}$$

We wish to solve for h the relaxation spectrum. We rearrange the above equation to obtain:

$$h = \mathscr{F}^{-1}\left[\frac{\hat{g}_1}{\hat{k_1}}\right]. \tag{1.37}$$

Now, we can attempt to calculate the Fourier transform of k_1:

$$\hat{k_1} = \mathscr{F}\left[1 + \tanh\left(\mu, \chi\right)\right]. \tag{1.38}$$

We note that neither 1 nor tanh are integrable on $\mathbb{R}$, and hence any attempt at calculating an expression for $\hat{k_1}$ would have to be in the sense of distributions.

If we combine the above we see that both the storage modulus and loss modulus equations reduce to the problem of solving:

$$h = \mathscr{F}^{-1}\left[\frac{\hat{g}}{\hat{k}}\right]. \tag{1.39}$$

For the case of the loss modulus;

$$\hat{k_2} = \pi \text{sech}\,\frac{p\pi}{2}. \tag{1.40}$$

However, as we have seen above, we can not obtain an expression for $\hat{k_1}$ in any classical sense. It is now that we understand why much of the inversion has been with the equation derived for the loss modulus.

In the next section we will demonstrate another result, which is seen in the work of Morgan [61], that favours inversion of the loss modulus over the storage modulus.

1.5.3 Newtonian Element of Fluid

We consider the two Fredholm integral equations that relate the relaxation spectrum $H(\tau)$ to the storage modulus $G^{'}(\omega)$ and loss modulus $G^{''}(\omega)$:

$$G^{'}(\omega) = \int_0^{\infty} \frac{\omega^2\tau^2}{1+\omega^2\tau^2}\frac{H(\tau)}{\tau}\,\mathrm{d}\tau, \tag{1.41}$$

$$G^{''}(\omega) = \int_0^{\infty} \frac{\omega\tau}{1+\omega^2\tau^2}\frac{H(\tau)}{\tau}\,\mathrm{d}\tau. \tag{1.42}$$

Now suppose that we write the relaxation spectrum $H(\tau) = \delta(\tau)$, where $\delta(\tau)$ is a

Dirac mass centered at zero. We can think of this Dirac mass as representing the purely Newtonian element of the fluid. Now if we replace the relaxation spectrum in equations (1.41) and (1.42) with $\delta(\tau)$, we get:

$$\begin{aligned} G'(\omega) &= \int_0^\infty \frac{\omega^2\tau^2}{(1+\omega^2\tau^2)} \frac{\delta(\tau)}{\tau} \, \mathrm{d}\tau \\ &= \int_0^\infty \frac{\omega^2\tau}{1+\omega^2\tau^2} \delta(\tau) \, \mathrm{d}\tau \\ &= \frac{0}{1+0} = 0, \end{aligned} \tag{1.43}$$

$$\begin{aligned} G''(\omega) &= \int_0^\infty \frac{\omega\tau}{(1+\omega^2\tau^2)} \frac{\delta(\tau)}{\tau} \, \mathrm{d}\tau \\ &= \int_0^\infty \frac{\omega}{1+\omega^2\tau^2} \delta(\tau) \, \mathrm{d}\tau \\ &= \frac{\omega}{1+0} = \omega, \end{aligned} \tag{1.44}$$

by properties of the Dirac mass (extending the integrals to the real line by defining them to be zero on the negative half-line).

What we can deduce from this is that the Fredholm integral equation relating the relaxation spectrum to the storage modulus in fact loses some information about the relaxation spectrum. The Newtonian part of the fluid as modelled by a Dirac mass relaxation spectrum, is not recoverable from the storage modulus. For this reason, one should not use equation (1.41) alone to recover the relaxation spectrum.

1.5.4 Linear Functional Strategy

An important analytical tool that Davies and Anderssen apply to the problem of recovering the relaxation spectrum is the Linear Functional Strategy. The linear functional strategy is a special case of a result of Goldberg where he considers a problem $s = Au$ where s is known, A is a bounded linear mapping and we wish to find u:

$$Au = \int_a^b k(x,\tau)\, u(\tau)\ \mathrm{d}\tau = s(x)\,. \tag{1.45}$$

Note that equation (1.45) is a Fredholm integral equation of the first kind i.e., the same type of problem that we introduced in equation (1.5). We define the data functional as:

$$L_\phi(s) = \int_a^b \phi(x)\, s(x)\ \mathrm{d}x = \langle \phi, s \rangle\,, \tag{1.46}$$

for some test function ϕ (where ϕ is the solution to equation (1.48)). Substituting equation (1.45) into equation (1.46) we obtain:

$$\begin{aligned} L_\phi(s) &= \int_a^b \phi(x) \left(\int_a^b k(x,\tau)\, u(\tau)\ \mathrm{d}\tau \right) \mathrm{d}x \\ &= \int_a^b \int_a^b \phi(x)\, k(x,\tau)\, u(\tau)\ \mathrm{d}\tau\, \mathrm{d}x. \end{aligned} \tag{1.47}$$

Assuming we can change the order of integration, that is:

$$\theta(\tau) = \int_a^b k(x,\tau)\,\phi(x)\,\mathrm{d}x = A^*\phi \tag{1.48}$$

is well defined, then:

$$\begin{aligned} L_\phi(s) &= \int_a^b \theta(\tau)\,u(\tau)\,\mathrm{d}\tau \\ &= L_\theta(u)\,. \end{aligned} \tag{1.49}$$

$L_\theta(u)$ is defined as the solution-functional.

Suppose we are trying to evaluate an expression in the form of the integral in equation (1.49), where u in unknown. The linear functional strategy allows us to instead evaluate L_ϕ where we know s and ϕ. We will see in the following calculations that we can obtain information about the relaxation spectrum H from a solution-functional $L_\theta(H)$ by evaluating a data functional instead. The details are to follow.

The problem of recovering the relaxation spectrum involves the inversion of Fredholm integral equations of the first kind:

$$\int_0^\infty k(\tau,\omega)\,H(\tau)\,\mathrm{d}\tau = f(\omega)\,, \tag{1.50}$$

where $k(\tau,\omega) = \omega^2\tau/\left(1+\omega^2\tau^2\right)$ for $f(\omega) = G'(\omega)$, where $G'(\omega)$ is the storage modulus and $k(\tau,\omega) = \omega/\left(1+\omega^2\tau^2\right)$ for $f(\omega) = G''(\omega)$, where $G''(\omega)$ is the loss modulus.

Now, the quantities η_{ab} and g_{ab}, known as the partial viscosity and elastic modulus respectively, are defined as:

$$\eta_{ab} = \int_a^b H\left(\tau\right)\,\mathrm{d}\tau, \tag{1.51}$$

$$g_{ab} = \int_a^b \frac{H\left(\tau\right)}{\tau}\,\mathrm{d}\tau. \tag{1.52}$$

Davies and Anderssen [20] note that in equations (1.51) and (1.52) they can make a and b arbitrarily close, so that in fact they are calculating an "average " of H over some small interval. They define two mean values for H as:

$$\overline{H}_{ab} = \frac{\eta_{ab}}{b-a}, \tag{1.53}$$

$$\overline{\overline{H}}_{ab} = \frac{g_{ab}}{\ln\left(b/a\right)}. \tag{1.54}$$

Davies and Anderssen note that both $\overline{H}_{ab}$ and $\overline{\overline{H}}_{ab}$ tend to the value $H\left(a\right)$ as b tends to a providing that $H\left(\tau\right)$ is continuous at $\tau = a$. More generally, we require H to have a Lebesgue point at a.

We recall that for a function f, x is a Lebesgue point in the domain of f if:

$$\lim_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y = 0,$$

where $B(x,r)$ is the ball centered at x with radius r, and $|B(x,r)|$ is the Lebesgue measure of that ball. The Lebesgue points of f are thus points where f does not oscillate too much, in an average sense. For an integrable function, almost every point is a Lebesgue point. Dodd [23] notes that if $H \in L^1(a,b)$ then H can be determined from taking limits as $b \to a$ of equations (1.53) and (1.54). In fact he also notes that to be able to apply the Linear Functional strategy to our problem, we should assume that H belongs to $L^1(a,b)$.

Now, equations (1.51) and (1.52) are both special cases of the solution-functional, which we can express in the form:

$$L_\theta(H) = \int_0^\infty \theta(\tau) \, H(\tau) \, \mathrm{d}\tau, \tag{1.55}$$

where $\theta(\tau)$ is equal to the box function $\beta_{ab}(\tau)$ for equation (1.51) and $\tau^{-1}\beta_{ab}(\tau)$ for equation (1.52).

We define the box function, $\beta_{ab}(\tau)$, as:

$$\beta_{ab}(\tau) = \begin{cases} 1, & \text{if } a \le \tau \le b, \\ 0, & \text{otherwise.} \end{cases}$$

The adjoint equation (to our original problem $AH = f$ in equation (1.50)) is defined as:

$$A^*\phi = \int_0^\infty k(\tau,\omega)\,\phi(\omega)\ \mathrm{d}\omega = \theta(\tau)\,, \tag{1.56}$$

and if this has a unique solution $\phi(\omega)$, then we can change the order of integration and:

$$\begin{aligned} L_\theta(H) &= \int_0^\infty \theta(\tau)\,H(\tau)\ \mathrm{d}\tau = \int_0^\infty \left\{ \int_0^\infty k(\tau,\omega)\,\phi(\omega)\ \mathrm{d}\omega \right\} H(\tau)\ \mathrm{d}\tau \\ &= \int_0^\infty \int_0^\infty k(\tau,\omega)\,H(\tau)\ \mathrm{d}\tau\phi(\omega)\ \mathrm{d}\omega = \int_0^\infty f(\omega)\,\phi(\omega)\ \mathrm{d}\omega \\ &= L_\phi(f)\,, \end{aligned} \tag{1.57}$$

where the integral $L_\phi(f)$ is called a data-functional. We deduce that when $\theta(\tau)$ is such that equation (1.56) has a unique solution $\phi(\omega)$, evaluating the data-functional, $L_\phi(f)$, is equivalent to evaluating the solution-functional, $L_\theta(H)$. This is called the linear functional strategy.

Another important thing to note is that in the work of Davies and Anderssen they extend the linear functional strategy beyond its original setting since, with regards to equation (1.56), the solution ϕ is in the sense of distributions.

1.5.5 Sampling Localisation

We review the sampling localisation theorems of Davies and Anderssen [20], where they make use of the linear functional strategy to obtain their results.

Davies and Anderssen define sampling localisation as:

> "Any situation where information about the relaxation spectrum on the time interval $a < \tau < b$ is determined completely by the values of the storage and loss moduli in some finite interval or intervals. "

In their paper [20], they claim that the earlier belief that information about the storage and loss moduli over an interval of frequencies $\omega_{min} < \omega < \omega_{max}$ determined the relaxation spectrum over the interval of relaxation times $(\omega_{max})^{-1} < \tau < (\omega_{min})^{-1}$ was incorrect. Previously, estimates of the relaxation spectrum outside the reciprocal range were discarded. However, Davies and Anderssen claim that the relaxation spectrum is determined on a shorter interval of relaxation times, corresponding to a larger interval of frequencies. They call this sampling localisation.

They deduce two sampling localisation theorems using the partial viscosity η_{ab} and elastic modulus g_{ab}, as defined in equations (1.51) and (1.52) (for proofs and more detail see [20]):

Theorem 1.1. *The First Sampling Localization Theorem*

The partial viscosity η_{ab} over the range of relaxation times $a < \tau < b$ is determined completely by the values of the storage modulus whose frequencies are in the range:

$$\frac{e^{-\pi/2}}{b} < \omega < \frac{e^{\pi/2}}{a}. \tag{1.58}$$

This theorem tells us that η_{ab} is determined by the storage modulus over a frequency interval with:

$$\frac{\omega_{max}}{\omega_{min}} = 10^{1.36}\left(\frac{b}{a}\right). \tag{1.59}$$

Therefore, a further 1.36 decades of frequency needs to be sampled outside of the traditional reciprocal range $b^{-1} < \omega < a^{-1}$.

Theorem 1.2. *The Second Sampling Localization Theorem*

If $1 < b/a < e^{\pi}$*, the partial viscosity* η_{ab} *over the range* $a < \tau < b$ *is determined completely by values of the loss modulus whose frequencies are in the range given by (1.58). However, if* $b/a > e^{\pi}$*, then* η_{ab} *is determined completely by values of the loss modulus whose frequencies lie in two distinct ranges*

$$\frac{e^{-\pi/2}}{b} < \omega < \frac{e^{\pi/2}}{b}, \quad \textit{and} \quad \frac{e^{-\pi/2}}{a} < \omega < \frac{e^{\pi/2}}{a}. \tag{1.60}$$

The two frequency ranges in the equation above are each of 1.36 decades. The second theorem tells us, therefore, that whatever the range $a < \tau < b$, no more than 2.73 decades are required to determine the partial viscosity η_{ab} from the loss modulus.

It would seem that the second part of the second theorem contradicts the first part.

However Davies and Anderssen note that the information contained by the loss modulus in overlapping frequency domains is cancelled out. In fact, the second part of the second theorem, i.e. the two distinct intervals, always gives us the values of ω that we require. However, when $b/a < e^{\pi}$ these two intervals overlap, so we simply take the union of the two intervals to give the one larger interval in the first part. The contradiction that Davies and Anderssen mention is simply the fact that both parts actually refer to the same interval. When they talk about overlapping frequency domains being cancelled out, all they are saying is the overlapping part is just considered once, i.e. the union of the two intervals.

Analogous calculations for the elastic modulus g_{ab} yield similar results.

1.5.6 Remarks

Both Renardy [69] and Macdonald [54] refer to Davies and Anderssen's sampling localization in their work.

Renardy [69] demonstrates in his work that by using a different kind of regularization (using polynomial approximation of a function in an exponentially weighted space, for more detail see [69]) to that of Davies and Anderssen [20] (who use regularization by a Gaussian) he can reconstruct the relaxation spectrum from data taken in any interval of time or frequency, however short and wherever located. His algorithm, however, turns out not to be practical. In a revised paper by Loy, Davies and Anderssen [51] they agree that Renardy's method has the advantage of being able to make use of properties of analytic functions. However, they note also the big disadvantage, that is, the severe ill-posedness of analytic continuation, which causes the generated approximations to behave increasingly badly as the interval on which they are assumed to be determined is extended further and further away from the limits predicted by Davies and Anderssen. Macdonald [54] compares two intervals for determining the relaxation spectrum, one being the reciprocal relaxation time and the other being the interval predicted by Davies and Anderssen in their sampling localization theorems. Macdonald plots exact points of a distribution of relaxation times along with inversion estimates. He also includes the magnitudes of the exact relative errors of the inversions and included on these are vertical lines which indicate the positions of the two types of sampling localization limits. Macdonald concludes that for the interval of frequencies predicted by Davies and Anderssen the relative estimation error is of the order of 1%, while for the reciprocal relaxation time interval the error is somewhat larger than 10%. This would suggest that Davies and Anderssens interval is more accurate. However, Macdonald goes on to say that there is clearly no abrupt or more rapid increase in error just outside the lower boundary of the interval predicted by Davies and Anderssen, as implied by their work [20]. Renardy [69] refers to the work of Macdonald in his paper by mentioning that Macdonald finds nothing special happening at the end points of the window predicted by Davies and Anderssen; rather, there is a gradual increase of error both inside and outside this window. In a revised paper by Loy, Davies and Anderssen [51] they interpret the results of Macdonald. They note that although Macdonald was able to recover approximations of the relaxation spectrum outside the limits predicted by Davies and Anderssen, his approximations displayed a similar loss of accuracy outside these limits as Renardy's approximations.

Clearly, there are conflicting opinions amongst these authors and one needs to clarify what exactly can be said about sampling localization. One of the aims of this thesis is to determine to what extent sampling localization has a sound theoretical basis.

1.6 Outline of the Thesis

In this chapter we have introduced the reader to the concepts of an inverse, ill-posed problem. We have demonstrated that the problem that we are trying to solve in this thesis is in fact an inverse, ill-posed problem. Hence, we will need to use regularisation methods in attempting to recover the relaxation spectrum. We have also described some of the background of linear viscoelasticity and what it means for a fluid to be viscoelastic. We have demonstrated how the Fredholm equations relating the relaxation spectrum to the storage and loss moduli are obtained and have briefly explained how one would obtain measurements of these quantities. A brief overview of the main attempts at recovering the continuous relaxation spectrum, or information about it, has been given. We demonstrated calculations for the storage modulus, similar to those that Loy, Newbury, Davies and Anderssen [50],[51], and Dodd [23] perform for the loss moduli, which confirms why the equation for the loss modulus is favoured over the equation for the storage modulus. We will go into greater detail with regards to the methods used by Loy, Newbury, Davies and Anderssen [50],[51], Dodd [23] and Renardy [69] in the following chapter and discuss the validity of these methods.

In Chapter 3 we make modifications to our problem in order to try and demonstrate a bound of exponential type for the inverse Fourier transform of the relaxation spectrum. We encounter problems that suggest that we cannot find such a bound. We will also perform calculations to demonstrate that Renardy's claim [69], that g is an analytic function is true and also why it cannot have compact support. Following this, we prove that the Paley-Wiener theorem cannot be applied to our problem, which suggest that the relaxation spectrum cannot have compact support.

In Chapter 4, we improve on the results of Chapter 3, by working in the space of tempered distributions. Using the Paley-Wiener-Schwartz theorem we are able to prove that the relaxation spectrum cannot have compact support in Schwartz space or in the L^p, L^q setting, which is the setting of the results of Loy, Newbury, Davies and Anderssen [50],[51], Dodd [23]. In the latter part of Chapter 4 we consider the type of functions that could satisfy our problem which helps us gain a better understanding of the problem.

In Chapter 5 we introduce a concept known as "compact essential numerical support" as being an interval of support for a function, outside which the function becomes insignificant. We demonstrate by examples that the relaxation spectrum does in fact become insignificant outside some closed interval. We demonstrate for certain functions that we can find a set outside which the supremum norm of the relaxation spectrum is negligible. We discuss how this concept might be adapted to other norms, for example L^p norms.

The final chapter collects together all the results and considers their strengths and weaknesses, and also outlines what further work could be done.

Chapter 2

Relaxation Spectrum Recovery - Formulation of Problem

2.1 Methods in Recovering the Relaxation Spectrum

The calculations that follow, involving the loss modulus, have been demonstrated in part in the work by Loy, Newbury, Anderssen and Davies [50], Dodd [23], Newbury [63] and Loy, Newbury, Anderssen and Davies [50]; however, we provide additional details and clarifications of the work that has been done.

After reducing the original Loss modulus Fredholm integral equation to equation (1.39) with k_2 as defined in (1.40), we define a space of functions: $F_{[\lambda,p]}$, appropriate to the study of the problem; those $g \in L^1(\mathbb{R})$ such that $\xi_\lambda \cdot \hat{g} \in L^p(\mathbb{R})$ for $1 < p \leq 2$ where $\xi_\lambda(r) = \cosh(\lambda r)$. From equations (1.39) and (1.40), we obtain:

$$h = \mathscr{F}^{-1}\left[\xi_\lambda \cdot \hat{g}\right]. \tag{2.1}$$

Hence, $\xi_\lambda \cdot \hat{g}$ is the Fourier transform of h; $\mathscr{F}(h)$ where h is the relaxation spectrum that we are trying to obtain.

We introduce a conjecture;

Conjecture 2.1. *Define;*

$$F_{[\lambda,p]} = \left\{g \in L^1(\mathbb{R}) \mid \xi_\lambda \cdot \hat{g} \in L^p(\mathbb{R})\right\}, \tag{2.2}$$

for $1 < p \leq 2$. *Then;*

$$h \equiv \mathscr{F}^{-1}\left[\hat{g} \cdot \xi_\lambda\right] \in L^q(\mathbb{R}) \tag{2.3}$$

and has compact support, where p *and* q *are conjugate exponents, i.e.* $p^{-1} + q^{-1} = 1$.

What we mean when we say that h has compact support is that the closure of $\{x : h(x) \neq 0\}$ is a compact subset of $\mathbb{R}$. For further detail see Jost [47].

Next, we introduce an important theorem that we will make use of in this thesis. It allows us to relate functions and their Fourier transforms in certain L^p spaces. For more detail, see Rudin [71].

Theorem 2.2. *Hausdorff-Young*

Let $f \in L^p(\mathbb{R})$ *where* $1 < p \leq 2$*,and let* q *denote the conjugate exponent of* p*, that is* $\frac{1}{p} + \frac{1}{q} = 1$*. Then;*

$$\left\|\hat{f}\right\|_q \leq \|f\|_p . \tag{2.4}$$

Now we are able to apply Theorem 2.2 to our function $\xi_\lambda \cdot \hat{g}$, since we have defined this as belonging to $L^p(\mathbb{R})$. Then its inverse Fourier transform, which is what we are attempting to calculate, will belong to $L^q(\mathbb{R})$. (We note later that the Fourier transform is a Hilbert space isomorphism on $L^2(\mathbb{R})$.) In keeping with the notation of the papers we reference, we will represent h (the relaxation spectrum) as κ_g, that is;

$$\kappa_g = \mathscr{F}^{-1}\left\{\hat{g}\left(r\right) \cdot \cosh(\lambda r)\right\}. \tag{2.5}$$

We are going to consider a variational formulation of equation (2.5), which is essentially: $\langle \kappa_g, f \rangle = \left\langle \hat{\kappa_g}, \hat{f} \right\rangle$ for $f \in L^p$.

We would like to obtain information about the properties of κ_g. We define a bilinear form:

$$\tilde{\kappa}_g\left(f\right) = \int_{-\infty}^{\infty} \hat{\kappa_g} \hat{f} \tag{2.6}$$

for $f \in L^p(\mathbb{R})$. We demonstrate this is a bounded linear mapping, and thus may be represented by an element of L^q, which we identify with κ_g.

We have that;

$$|\tilde{\kappa}_g(f)| = \left|\int_{-\infty}^{\infty} \xi_\lambda \hat{g} \cdot \hat{f}\right| = \left|\left\langle \xi_\lambda \hat{g}, \hat{f}\right\rangle\right| \leq \|\xi_\lambda \hat{g}\|_p \left\|\hat{f}\right\|_q, \tag{2.7}$$

where we have applied Hölder's inequality.

Furthermore, we apply the Hausdorff-Young Theorem, which gives us;

$$|\tilde{\kappa}_g(f)| \leq \|\xi_\lambda \hat{g}\|_p \left\|\hat{f}\right\|_q \leq \|\xi_\lambda \hat{g}\|_p \|f\|_p. \tag{2.8}$$

That is, $\tilde{\kappa}_g$ satisfies the condition of a bounded linear operator on $L^p(\mathbb{R})$. Alternatively, we can say it belongs to the dual space $(L^p(\mathbb{R}))^* \cong L^q(\mathbb{R})$. We can identify κ_g with an element of $L^q(\mathbb{R})$, which justifies the connection between (2.5) and (2.6).

We write

$$\tilde{\kappa}_g(f) = \int_{-\infty}^{\infty} \hat{\kappa_g} \hat{f} = \int_{-\infty}^{\infty} \kappa_g f = \langle \kappa_g, f\rangle. \tag{2.9}$$

2.1.1 Interchanging Order of Integration

We can express the bilinear form in the following way:

$$\begin{aligned} \langle \kappa_g, f \rangle &= \int_{-\infty}^{\infty} \xi_\lambda(r)\, \hat{g}(r)\, \hat{f}(r)\, \mathrm{d}r \qquad (2.10) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cosh(\lambda r)\, (g * f)(s)\, e^{-irs}\, \mathrm{d}s\, \mathrm{d}r. \qquad (2.11) \end{aligned}$$

By way of explanation, equation (2.10) makes sense as $\xi_\lambda(r)\, \hat{g}(r)\, \hat{f}(r) \in L^1(\mathbb{R})$, and we have used the fact that $\widehat{g * f} = \hat{g}\hat{f}$ (which is justified by the methods of Rudin [71] (Theorem 8.14, exercise 4, p. 174, Theorem 9.2)). We wish to change the order of integration in equation (2.11); we recall the definition of a complete measure space, and state Fubini's Theorem in such a setting.

Definition 2.3. *Measure Space*

A measure space, $(\Omega, \mathcal{A}, \mu)$, is said to be complete if all subsets of $\mathcal{A}$-measurable sets of μ-measure zero are also $\mathcal{A}$-measurable. That is, if $A \in \mathcal{A}$ and $\mu(A) = 0$, then $B \in \mathcal{A}$ for all $B \subset A$.

Theorem 2.4. *Fubini's Theorem*

Let X and Y be complete measure spaces, and;

$$\int_{X \times Y} |f(x, y)|\, \mathrm{d}(x, y) < \infty, \qquad (2.12)$$

then;

$$\int_X \left(\int_Y f(x,y)\, \mathrm{d}y \right) \mathrm{d}x = \int_Y \left(\int_X f(x,y)\, \mathrm{d}x \right) \mathrm{d}y \tag{2.13}$$

$$= \int_{X \times Y} f(x,y)\, \mathrm{d}(x,y)\,. \tag{2.14}$$

For more detail see Rudin [71]. Referring back to equation (2.10):

$$\langle \kappa_g, f \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cosh(\lambda r)\,(g * f)(s)\, e^{-irs}\, \mathrm{d}s\, \mathrm{d}r. \tag{2.15}$$

We wish to change the order of integration in equation (2.15), integrating with respect to r first. Unfortunately the Fourier transform of cosh does not fit our L^p theory. This problem is overcome by introducing a mollifier.

2.1.2 Mollification

We introduce a Mollifier;

$$\exp\left\{ \frac{-\epsilon^2 r^2}{2} \right\}.$$

For a fixed $\epsilon \neq 0$, the mollifier $\exp\{-\epsilon^2 r^2/2\} \to 0$ as $|r| \to \infty$ and the rate of decay is faster than cosh diverges to infinity as $|r| \to \infty$.

We include this mollifier in our original bilinear form and define a new mollified function.

For $\epsilon \neq 0$, define $\tilde{\kappa}_{g,\epsilon} : L^p(\mathbb{R}) \to \mathbb{R}$ by

$$\tilde{\kappa}_{g,\epsilon}(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cosh(\lambda r)\exp\left\{\frac{-\epsilon^2 r^2}{2}\right\}(g*f)(s)\,e^{-irs}\,\mathrm{d}s\,\mathrm{d}r. \tag{2.16}$$

By analogous reasoning to the above, $\tilde{\kappa}_{g,\epsilon}(f) = \langle \kappa_g(\epsilon), f\rangle$ for some $\kappa_{g,\epsilon} \in L^q(\mathbb{R})$.

We justify the use of this mollifier by noting that as $\epsilon \to 0$, the mollifier tends to 1. This mollifier 'controls' the cosh function. With regards to r the integral in equation (2.16) is finite and making use of Fubini's theorem allows us to interchange the order of integration.

We manipulate equation (2.16) and make use of the pdf of a Normal distribution (see Fourier Transform 3 in the appendix), to obtain;

$$\begin{aligned}
\langle \kappa_g(\epsilon), f\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cosh(\lambda r)\exp\left\{\frac{-\epsilon^2 r^2}{2}\right\} e^{-irs}(g*f)(s)\,\mathrm{d}r\,\mathrm{d}s \\
&= \int_{-\infty}^{\infty} \frac{\sqrt{2\pi}}{\epsilon}\exp\left\{\frac{\lambda^2 - s^2}{2\epsilon^2}\right\}\cos\left(\frac{\lambda s}{\epsilon^2}\right)(g*f)(s)\,\mathrm{d}s \\
&= \int_{-\infty}^{\infty} W_\epsilon(s)(g*f)(s)\,\mathrm{d}s, \qquad (2.17)
\end{aligned}$$

where

$$W_\epsilon(s) = \frac{\sqrt{2\pi}}{\epsilon} \exp\left\{\frac{\lambda^2 - s^2}{2\epsilon^2}\right\} \cos\left(\frac{\lambda s}{\epsilon^2}\right). \tag{2.18}$$

2.1.3 Taking Limits of Mollified Bilinear Form

Let $\delta > 0$ be given, we show;

1. $\lim_{\epsilon\to 0} \int_{|s|>\lambda+\delta} W_\epsilon(s)\,(g * f)(s)\,\mathrm{d}s = 0$,

2. $\lim_{\epsilon\to 0} \langle \kappa_g(\epsilon), f\rangle = \tilde{\kappa}_g(f) \equiv \langle \kappa_g, f\rangle$,

and deduce from these that:

$$\langle \kappa_g, f\rangle = \lim_{\epsilon\to 0} \int_{|s|\le\lambda+\delta} W_\epsilon(s)\,(g * f)(s).$$

We define an important theorem, statement taken from Jost [47].

Theorem 2.5. *Dominated Convergence Theorem*

Let $f_n : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ be a sequence of integrable functions which converge on $\mathbb{R}$ pointwise almost everywhere to a function $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$. Moreover, assume that there is an integrable function $G : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ with:

$$|f_n| \le G \text{ for all } n \in \mathbb{N}.$$

Then f is integrable and

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\,\mathrm{d}x = \int_{\mathbb{R}} \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\,\mathrm{d}x. \tag{2.19}$$

We would like to apply this theorem to our mollified function to show that:

$$\int_{\mathbb{R}} W\left(g*f\right) = \lim_{\epsilon\to 0}\int \mathbb{R} W_\epsilon\left(g*f\right) = \int \mathbb{R}\lim_{\epsilon\to 0} W_\epsilon\left(g*f\right), \tag{2.20}$$

where W is the pointwise limit, almost everywhere (for $|s| > \lambda$), of W_ϵ (as $\epsilon \to 0$).

In considering $|W_\epsilon(s)|$, Dodd [23] considers two cases. We split the integral, writing:

$$\begin{aligned}\int W_\epsilon(s)\left(g*f\right)(s)\ \mathrm{d}s &= \int_{\{s:|(g*f)(s)|<1\}} W_\epsilon(s)\left(g*f\right)(s)\ \mathrm{d}s \\ &+ \int_{\{s:|(g*f)(s)|\geq 1\}} W_\epsilon(s)\left(g*f\right)(s)\ \mathrm{d}s. \end{aligned}\tag{2.21}$$

We demonstrate calculations for the following cases:

- Case 1: $|(g*f)(s)| < 1$
- Case 2: $|(g*f)(s)| \geq 1$

Furthermore, consider: $|s| > \lambda + \delta$, then $s^2 > (\lambda+\delta)^2 = \lambda^2 + 2\lambda\delta + \delta^2$.

That is: $s^2 - 2\lambda\delta > \lambda^2 + \delta^2$ which implies that $s^2 > \lambda^2 + \delta^2$, which we will make use of in the following calculations.

2.1.4 Case 1: $|(g*f)(s)| < 1$

We evaluate the absolute value of the integrand $W_\epsilon(s)\left(g*f\right)(s)$;

$$
\begin{aligned}
\left|W_\epsilon(s)(g*f)(s)\right| &= \left|\frac{1}{\epsilon}\exp\left\{\frac{\lambda^2-s^2}{2\epsilon^2}\right\}\cos\left(\frac{\lambda s}{\epsilon^2}\right)(g*f)(s)\right| \\
&\leq \left|\frac{1}{\epsilon}\exp\left\{\frac{\lambda^2-s^2}{2\epsilon^2}\right\}\right| \\
&= \left|\frac{1}{\epsilon}\exp\left\{\frac{\lambda^2}{2\epsilon^2}+s^2\left(1-\frac{1}{2\epsilon^2}\right)\right\}\exp\left\{-s^2\right\}\right|.
\end{aligned}
$$

We note that $s^2 > \lambda^2 + \delta^2$ and for $\epsilon < 1/\sqrt{2}$ we have $(1 - 1/2\epsilon^2) < 0$. Now:

$$
\begin{aligned}
\left|W_\epsilon(s)(g*f)(s)\right| &< \left|\frac{1}{\epsilon}\exp\left\{\frac{\lambda^2}{2\epsilon^2}+\left(\lambda^2+\delta^2\right)\left(1-\frac{1}{2\epsilon^2}\right)\right\}\exp\left\{-s^2\right\}\right| \\
&= \left|\frac{1}{\epsilon}\exp\left\{\delta^2\left(1-\frac{1}{2\epsilon^2}\right)\right\}\exp\left\{\lambda^2\right\}\exp\left\{-s^2\right\}\right|.
\end{aligned}
$$

Exponential decrease dominates algebraic increase as $\epsilon \to 0$, therefore there exists $M > 0$ such that

$$
\frac{1}{\epsilon}\exp\left\{\delta^2\left(1-\frac{1}{2\epsilon^2}\right)\right\} < \mathrm{M},
$$

for all $\epsilon > 0$. Hence:

$$
\left|W_\epsilon(s)(g*f)(s)\right| < \mathrm{M}\left|\exp\left\{\lambda^2\right\}\exp\left\{-s^2\right\}\right| = G_1(s).
$$

Now, G_1 is integrable, and will act as a dominator when we apply the Dominated Convergence Theorem.

2.1.5 Case 2: $|(g * f)(s)| \geq 1$

First we note that $|(g * f)(s)| \geq 1 \Rightarrow |(g * f)(s)| \leq |(g * f)(s)|^p$. Now;

$$\begin{aligned} |W_\epsilon(s)(g * f)(s)| &= \left| \frac{1}{\epsilon} \exp\left\{ \frac{\lambda^2 - s^2}{2\epsilon^2} \right\} \cos\left(\frac{\lambda s}{\epsilon^2} \right)(g * f)(s) \right| \\ &\leq \left| \frac{1}{\epsilon} \exp\left\{ \frac{\lambda^2 - s^2}{2\epsilon^2} \right\} \right| |(g * f)(s)|^p . \end{aligned}$$

We perform similar steps as for Case 1 to get:

$$\begin{aligned} |W_\epsilon(s)(g * f)(s)| &< \mathrm{M} \left| \exp\left\{ \lambda^2 \right\} \exp\left\{ -s^2 \right\} \right| |(g * f)(s)|^p \\ &\leq M \exp\left\{ \lambda^2 \right\} |(g * f)(s)|^p = |G_2(s)| . \end{aligned}$$

Finally, we need to check that the second dominator is integrable. Given that $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ it follows from Rudin [71], (exercise 4, p174) that $(g * f)(s) \in L^p(\mathbb{R})$, which implies that $|(g * f)(s)|^p \in L^1(\mathbb{R})$. We have shown that G_2 is integrable.

We may now apply the Dominated Convergence Theorem. For both cases we have:

$$\int_{\mathbb{R}} \lim_{\epsilon \to 0} W_\epsilon(s)(g * f)(s) = \int_{\mathbb{R}} \lim_{\epsilon \to 0} \frac{\sqrt{2\pi}}{\epsilon} \exp\left\{ \frac{\lambda^2 - s^2}{2\epsilon^2} \right\} \cos\left(\frac{\lambda s}{\epsilon^2} \right)(g * f)(s) . \tag{2.22}$$

Given that $\lambda^2 - s^2 < 0$ then

$$\frac{1}{\epsilon} \exp\left\{ \frac{\lambda^2 - s^2}{2\epsilon^2} \right\} \to 0 \text{ as } \epsilon \to 0 \tag{2.23}$$

and noting that $(g * f) \in L^p(\mathbb{R})$, $(g * f)(s)$ is finite for a.e. s, so we have:

$$\lim_{\epsilon \to 0} W_\epsilon(s)(g * f)(s) = 0 \text{ for a.e. } s.$$

For both cases we have:

$$\lim_{\epsilon \to 0} \int W_\epsilon(s)(g * f)(s) = \int \lim_{\epsilon \to 0} W_\epsilon(s)(g * f)(s) = 0. \tag{2.24}$$

Hence

$$\lim_{\epsilon \to 0} \langle \kappa_g(\epsilon), f \rangle = \lim_{\epsilon \to 0} \int_{|s| \leq \lambda + \delta} W_\epsilon(s)(g * f)(s). \tag{2.25}$$

We have demonstrated the first of the two limits that we wanted to show, now we consider the second; $\lim_{\epsilon \to 0} \langle \kappa_g(\epsilon), f \rangle = \langle \kappa_g, f \rangle$.

Now,

$$\langle \kappa_g - \kappa_g(\epsilon), f \rangle = \int_{-\infty}^{\infty} \xi_\lambda(r)\, \hat{g}(r)\, \hat{f}(r) \left[1 - e^{\frac{-\epsilon^2 r^2}{2}}\right] \mathrm{d}r \tag{2.26}$$

and

$$\left|\xi_\lambda\left(r\right)\hat{g}\left(r\right)\hat{f}\left(r\right)\left[1-\exp\left\{-\epsilon^2 r^2/2\right\}\right]\right|\leq\left|\xi_\lambda\left(r\right)\hat{g}\left(r\right)\hat{f}\left(r\right)\right|,$$

for every ϵ, with $\xi_\lambda \hat{g}\hat{f} \in L^1(\mathbb{R})$.

Noting that $\xi_\lambda\left(r\right)\hat{g}\left(r\right)\hat{f}\left(r\right)\left[1-\exp\left\{-\epsilon^2 r^2/2\right\}\right]\to 0$ as $\epsilon \to 0$ for a.e. r we have:

$$\left\langle\kappa_g, f\right\rangle-\lim_{\epsilon\to 0}\left\langle\kappa_g\left(\epsilon\right), f\right\rangle=\lim_{\epsilon\to 0}\left\langle\kappa_g-\kappa_g\left(\epsilon\right), f\right\rangle=0 \tag{2.27}$$

by the Dominated Convergence Theorem.

Combining the above;

$$\lim_{\epsilon\to 0}\left\langle\kappa_g\left(\epsilon\right), f\right\rangle=\left\langle\kappa_g, f\right\rangle. \tag{2.28}$$

Combining our two results we obtain:

$$\left\langle\kappa_g, f\right\rangle=\lim_{\epsilon\to 0}\int_{|s|\leq\lambda+\delta}\frac{\sqrt{2\pi}}{\epsilon}\exp\left\{\frac{\lambda^2-s^2}{2\epsilon^2}\right\}\cosh\left(\frac{\lambda s}{\epsilon^2}\right)\left(g*f\right)\left(s\right) \tag{2.29}$$

for any $\delta > 0$.

2.2 Conclusions

The only part of the integrand in equation (2.29) that involves the functions g and f is the convolution of g with f. We consider this;

$$(g * f)(s) = \int_{-\infty}^{\infty} g(t) f(s-t) \, \mathrm{d}t. \tag{2.30}$$

In the work of Davies, Anderssen, Loy and Newbury [50] and Dodd [23] they make the additional assumption that g has compact support in the interval $[a, b]$. Therefore, this integral reduces to;

$$\int_{a}^{b} g(t) f(s-t) \, \mathrm{d}t. \tag{2.31}$$

They define $f \in L^p(\mathbb{R})$ supported on the interval $\mathbb{R} \backslash [-\lambda - \delta - b, \lambda + \delta - a]$. Since the integral in equation (2.29) is defined for values of s such that $|s| \leq \lambda + \delta$, it follows that equation (2.31) and hence equation (2.29) are equal to zero for f supported on the interval $\mathbb{R} \backslash [-\lambda - \delta - b, \lambda + \delta - a]$. They deduce from this that f has no effect on the bilinear form in the interval $\mathbb{R} \backslash [-\lambda - \delta - b, \lambda + \delta - a]$. Hence the effect must be on $[-\lambda - \delta - b, \lambda + \delta - a]$. They interpret this as κ_g being supported on the interval $[-\lambda - \delta - b, \lambda + \delta - a]$.

2.2.1 Contradicting Results

In a paper by Renardy [69], he evaluates the relaxation spectrum, and refers to the previous work by Loy, Newbury, Anderssen and Davies. He brings to light a very important fact that was overlooked in previous work.

We refer back to Loy, Newbury, Anderssen and Davies' notation: $g \in F_{[\lambda,p]}$ if $g \in L^1(\mathbb{R})$, $\xi_\lambda \cdot \hat{g} \in L^p(\mathbb{R})$ and furthermore g is assumed to have compact support. However, Renardy points out that the assumption $\xi_\lambda \cdot \hat{g} \in L^p(\mathbb{R})$ makes g an analytic function. This is incompatible with the assumption that g has compact support (except for the trivial case $g = 0$).

Therefore, it follows that much of the evaluation and hence the results of the papers of Dodd [23], Loy, Newbury, Anderssen and Davies [50], are only valid for g equal to the zero function.

However, the results of Renardy's paper [69] motivated Loy, Davies and Anderssen to write a revised paper [51] addressing their methods for obtaining an interval of support for the relaxation spectrum.

2.3 Revised Calculations

The important argument that Renardy writes about in his paper [69], is that for the space defined in Conjecture 2.1, the functions in this space are analytic and so cannot have compact support (except for the trivial case).

We demonstrate in Chapter 3 that g, as defined in Conjecture 2.1, is an analytic function, and that the only possible g with compact support is the zero function.

2.3.1 Compact Support

Now, Loy, Davies and Anderssen [51] work with the same bilinear form as in their previous paper [50], and make similar calculations up to the point of obtaining equation (2.29):

$$\langle \kappa_g, f \rangle = \lim_{\epsilon \to 0} \int_{|s| \leq \lambda + \delta} \frac{\sqrt{2\pi}}{\epsilon} \exp\left\{ \frac{\lambda^2 - s^2}{2\epsilon^2} \right\} \cosh\left(\frac{\lambda s}{\epsilon^2} \right) (g * f)(s) \tag{2.32}$$

for $\delta > 0$.

Similar to previous calculations, we note that the only part of the integral above that contains the functions f and g is the convolution of g with f

$$(g * f)(s) = \int_{-\infty}^{\infty} g(t) f(s - t) \, \mathrm{d}t. \tag{2.33}$$

Loy, Davies and Anderssen [51] introduce a function c, compactly supported on $[a, b]$. Instead of evaluating the integral $\kappa_g(f)$ by considering the convolution g with f, they consider the convolution $g + c$ with f.

For g as defined in Conjecture 2.1, define $f \in L^p(\mathbb{R})$, supported on $\mathbb{R} \backslash [-\lambda - \delta - b, \lambda + \delta - a]$. Choose an f such that $f(t) = 0$ for $t \in [-\lambda - \delta - b, \lambda + \delta - a]$ and let c be an integrable function supported on $[a, b]$. Then;

$$\begin{aligned} (\{g+c\} * f)(s) &= \int_{-\infty}^{\infty} (g+c)(t)\, f(s-t)\ \mathrm{d}t \\ &= \int_{-\infty}^{\infty} g(t)\, f(s-t)\ \mathrm{d}t + \int_{-\infty}^{\infty} c(t)\, f(s-t)\ \mathrm{d}t. \end{aligned}$$

Now, c is defined with compact support in $[a, b]$ and f is defined with compact support in $\mathbb{R}\backslash[-\lambda-\delta-b, \lambda+\delta-a]$. Since we are integrating t over $[a, b]$, providing $|s| \leq \lambda+\delta$, then $f(s-t) = 0$. So, we can conclude;

$$\begin{aligned} (\{g+c\} * f)(s) &= \int_{-\infty}^{\infty} g(t)\, f(s-t)\ \mathrm{d}t + \int_{a}^{b} c(t)\, f(s-t)\ \mathrm{d}t \\ &= \int_{-\infty}^{\infty} g(t)\, f(s-t)\ \mathrm{d}t + 0 = (g * f)(s). \end{aligned} \tag{2.34}$$

That is, the effect of c on $\langle \kappa_g, f\rangle$ is zero provided that $f \in L^p(\mathbb{R})$ is supported outside the interval $[-\lambda-\delta-b, \lambda+\delta-a]$. The interpretation is that the behaviour of g on the interval $[a, b]$ has no effect on $\langle \kappa_g, f\rangle$ for f as defined above.

We conclude from the result above that the effect on $\langle \kappa_g, f\rangle$ must be supported in $[-\lambda-b, \lambda-a]$. Note that this is the same interval of support that Loy, Newbury, Anderssen and Davies had in their previous work [50]. However, we note that this is support in a weak sense since, essentially, what they have shown is that:

$$\langle \kappa_{g+c}, f\rangle = \langle \kappa_g, f\rangle \tag{2.35}$$

for $f \in L^p(\mathbb{R})$, supported on $\mathbb{R}\backslash[-\lambda-\delta-b, \lambda+\delta-a]$. Note, however, that $g+c \notin F_{[\lambda,p]}$.

In the subsequent chapters we prove that for the space of functions we are working in, Conjecture 2.1 is incorrect.

2.4 Another Approach to Recovering the Relaxation Spectrum

We make a few remarks with regards to a paper by Renardy, [69], in which he also uses regularisation methods to try and obtain information about the relaxation spectrum.

Renardy introduces the problem of obtaining the relaxation spectrum $\mu(\lambda)$ from the relaxation modulus $G(t)$ via a Laplace transform:

$$G(t) = \int_0^\infty \mu(\lambda)\, e^{-\lambda t}\, \mathrm{d}\lambda. \tag{2.36}$$

This is similar to the expression we introduced in equation (1.16), that is, an expression for the relaxation modulus defined as a Laplace transform of the relaxation spectrum. We note that there are slight differences in the two expressions. Comparing equation (2.36) with equation (1.16) we deduce that H is related to μ in the following relationship: $\tau H(\tau) = \mu(\tau^{-1})$. Hence, the two expressions for the relaxation spectrum, H and μ are not interchangeable. Recovering the relaxation spectrum from equation (2.36) is also an ill-posed problem. Hence, Renardy must use regularisation methods to obtain an expression for μ.

Renardy performs similar steps to those of Davies, Anderssen, Loy and Newbury [50], whereby he makes a change of variables in his integral equation. He proceeds by calculating the Fourier transform of $\Phi(v)$, his new representation of $G(t)$. This allows him to obtain an expression relating the new representation of the relaxation spectrum with the new representation of $G(t)$.

Renardy introduces a mollifier similar to the mollifier in the work of Davies, Anderssen, Loy and Newbury [50], that is, a Gaussian function. The only difference is that Renardy mollifies the Relaxation spectrum representation directly. Renardy mentions why he uses a Gaussian to regularise this ill-posed problem instead of other possible regularisation methods, namely Tikhonov regularisation and also truncation of the function. The advantage of using a Gaussian is that its inverse Fourier transform is also a Gaussian and hence, positive. This ensures that if the relaxation spectrum representation $\Psi(\sigma)$ is positive, then so is the mollified version of relaxation spectrum representation $\Psi_\epsilon(\sigma)$. After mollifying the relaxation spectrum representation $\Psi_\epsilon(\sigma)$, Renardy gives the steps needed to then obtain $\Psi_\epsilon(\sigma)$ from $\Phi(v)$ the representation for $G(t)$. He does not perform the calculations himself nor does he apply his method to data, so it is unknown how applicable his method is.

Another result of Davies and Anderssen [20] that Renardy [69] refers to in his work is *sampling localization*, which we introduced in Chapter 1. Renardy [69] disagrees with their claim and demonstrates that he can, in principle, obtain a regularised relaxation spectrum from data for $G(t)$ (Relaxation modulus) when t is limited to any interval. He does this by introducing a test function $\chi(u)$ such that χ is of class C^{∞}, $0 \leq \chi(u)$ everywhere, $\chi(u) = 0$ for $|u| > 1$ and

$$\int_{-\infty}^{\infty} \chi(u)\, \mathrm{d}u = 1. \tag{2.37}$$

Renardy notes that the Fourier transform $\hat{\chi}(p)$ tends to zero as $|p| \to \infty$ at a much faster rate than any reciprocal power of $|p|$. We note that this is the same type of decay that we have for functions belonging to the Schwartz space. That is, the space of rapidly decaying test functions such that the function and all its derivatives exist everywhere and go to zero at infinity faster than any inverse power. Renardy defines $\chi_{\epsilon}(u) = \chi(v/\epsilon)$. We note that the Fourier transform of $\chi_{\epsilon}(u)$ is $\hat{\chi}(v\epsilon)$.

He defines the space X_{δ} of all continuous functions on the real line for which

$$\lim_{x \to \pm\infty} e^{-\delta|x|} |f(x)| = 0, \tag{2.38}$$

with norm

$$\|f\|_{\delta} = \max_{x} e^{-\delta|x|} |f(x)|. \tag{2.39}$$

Note that the polynomials form a dense subset of X_{δ} for any $\delta > 0$. This can be demonstrated using results of Bernstein's Approximation Problem, an extension of the Weierstrass approximation theorem to the whole of the real line. Bernstein considers the following problem (for details see Lubinsky [52]):

Let $W : \mathbb{R} \to [0, 1]$ be measurable. When is it true that for every continuous $f : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{|x| \to \pm\infty} (fW)(x) = 0,$$

there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ with

$$\lim_{n \to \infty} \|(f - P_n)\, W\|_{L^\infty} = 0?$$

If true we then say that the polynomials are dense.

Lubinsky notes (corollary 1.5 of [52]) that this problem is satisfied if and only if $W_\alpha(x) = \exp\{-|x|^\alpha\}$ for $\alpha \geq 1$. If we compare this with the weight in equation (2.38), we can see that Bernstein's Approximation problem is satisfied and hence the polynomials are dense in X_δ.

Renardy notes that for a sufficiently small δ his representation of the relaxation spectrum, $\Psi_\epsilon(\sigma)$ can be approximated by the polynomials in the space X_δ.

Renardy uses a polynomial $P_\epsilon(\nu)$ to approximate the function $f_\epsilon(\nu)$, where $f_\epsilon(\nu)$ is such that: $\hat{\Psi}_\epsilon(\nu) = e^{-i\alpha\nu} f_\epsilon(\nu)\, \hat{\Phi}(\nu)$. We remind the reader that Ψ_ϵ is a mollified representation of μ the relaxation spectrum and Φ is a representation of G, the relaxation modulus.

Renardy defines:

$$Q_\epsilon(\nu) = P_\epsilon(\nu)\,\hat{\chi}(\epsilon\nu),$$

allowing him to write:

$$\hat{\Psi}_\epsilon(\nu) = Q_\epsilon(\nu)\,\hat{\Phi}(\nu).$$

Renardy takes inverse Fourier transforms of both sides and forms a convolution (making use of a result we introduced at the beginning of the chapter, see Rudin [71]) to give:

$$\Psi_\epsilon(\sigma) = \int_{-\infty}^{\infty} S_\epsilon(\sigma+\tau)\,\Phi(\tau)\;\mathrm{d}\tau. \tag{2.40}$$

Where S_ϵ is the inverse Fourier transform of Q_ϵ. Renardy notes that in the limit as $\epsilon \to 0$, we have that $\Psi_\epsilon(\sigma) \to \Psi_\epsilon(\sigma-\alpha)$. Furthermore, we note that:

$$S_\epsilon(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_\epsilon(\nu)\,\hat{\chi}(\epsilon\nu)\;\mathrm{d}\nu. \tag{2.41}$$

Since P_ϵ is a polynomial, we know from properties of Fourier transforms that the LHS of equation (2.41) will be a linear combination of derivatives of $\chi(\sigma/\epsilon)$. It follows that the support of S_ϵ is contained in the interval $(-\epsilon, \epsilon)$. Hence it follows from equation (2.40) that we can construct an apprximation to $\Psi(\sigma - \alpha)$ which uses only the values of Φ in the interval $(-\sigma - \epsilon, -\sigma + \epsilon)$.

However, Renardy concludes that in practice, reconstructing the relaxation spectrum from data in any arbitrary interval turns out to be quite difficult. His method of approximations by polynomials in the space X_δ is very poor in practice and he admits that it is unlikely that it can be applied to real data. He demonstrates this with an example.

2.5 Sampling Localisation and Linear Functional Strategy

It would appear from the work of Renardy [69], that the sampling localization theorems of Davies and Anderssen [20] are flawed. We consider their calculations in the following section to try and clarify what they have demonstrated.

2.5.1 Linear Functional Strategy

The problem of recovering the relaxation spectrum involves the inversion of Fredholm integral equations of the first kind:

$$\int_0^\infty k(\tau, \omega) H(\tau)\, \mathrm{d}\tau = f(\omega), \tag{2.42}$$

where $k(\tau, \omega) = \omega^2 \tau / \left(1 + \omega^2 \tau^2\right)$ for $f(\omega) = G'(\omega)$, where $G'(\omega)$ is the storage modulus and $k(\tau,\omega) = \omega/\left(1+\omega^2\tau^2\right)$ for $f(\omega) = G^{''}(\omega)$, where $G^{''}(\omega)$ is the loss modulus.

The idea behind this method of Davies and Anderssen [20], is to calculate the relaxation spectrum from the two quantities η_{ab} and g_{ab} defined in equations (1.51) and (1.52) respectively, by defining the following relationships between the relaxation spectrum $H(\tau)$ (providing H is integrable) and η_{ab} and g_{ab}:

$$H(a) = \lim_{b\to a} \frac{\eta_{ab}}{b-a}, \tag{2.43}$$

$$H(e^a) = \lim_{b\to a} \frac{g_{ab}}{\ln(b/a)}. \tag{2.44}$$

Now, if we refer to the linear functional strategy of Anderssen [6], we can note that equations (1.51) and (1.52) are both special cases of the solution-functional.

2.5.2 Sampling Localisation

We demonstrate how the linear functional strategy plays a key role in deducing the sampling localisation theorems. We demonstrate calculations for the partial viscosity but note that analogous calculations yield similar results for the elastic modulus.

We would like to write the partial viscosity, as defined in equation (1.51) as a data-functional of the storage modulus $G^{'}$, in the form:

$$\eta_{ab} = \int_0^\infty \phi'(\omega) G^{'}(\omega) \, \mathrm{d}\omega, \tag{2.45}$$

where $\phi'(\omega)$ satisfies (as a result of equation (1.48)):

$$\int_0^\infty \frac{\omega^2 \tau}{1+\omega^2\tau^2} \phi^{'}(\omega) \, \mathrm{d}\omega = \beta_{ab}(\tau). \tag{2.46}$$

Making a change of variables: $\omega = e^{-s}$ and $\tau = e^t$ and letting $\omega^2\phi'(\omega) = \varphi^{'}(\omega) = \Psi^{'}(s)$ and $\beta_{ab}(e^t) = B_{ab}(t)$, equation (2.46) becomes:

$$\int_{-\infty}^\infty \operatorname{sech}(s-t) \Psi^{'}(s) \, \mathrm{d}s = B_{ab}(t). \tag{2.47}$$

This is the definition of the convolution of sech with $\Psi^{'}$.

That is:

$$\operatorname{sech} * \Psi^{'} = B_{ab}. \tag{2.48}$$

We note that the calculations that follow are formal calculations and we will assume for the moment that we can perform these steps without justification.

Taking Fourier transforms of both sides and assuming that $\hat{\Psi}^{'}$ and B_{ab} are integrable functions we can write the Fourier transform of a convolution as a product of Fourier transforms, to give:

$$\begin{aligned} \mathscr{F}\left[\text{sech} * \Psi^{'}\right] &= \mathscr{F}\left[B_{ab}\right] \\ \mathscr{F}\left[\text{sech}\right]\mathscr{F}\left[\Psi^{'}\right] &= \mathscr{F}\left[B_{ab}\right]. \end{aligned} \tag{2.49}$$

The Fourier transform of a sech function gives another sech function and, assuming that the Fourier transform of $\Psi^{'}$ exists we can write equation (2.49) as:

$$\text{sech}\left(\frac{\pi p}{2}\right)\hat{\Psi}'\left(p\right) = \hat{B}_{ab}\left(p\right). \tag{2.50}$$

Rearranging, we obtain;

$$\begin{aligned} \hat{\Psi}^{'}\left(p\right) &= \hat{B}_{ab}\left(p\right)\cosh\left(\frac{\pi p}{2}\right) \\ \Psi^{'}\left(s\right) &= \mathscr{F}^{-1}\left[\hat{B}_{ab}\left(p\right)\cosh\left(\frac{\pi p}{2}\right)\right]. \end{aligned} \tag{2.51}$$

Now, B_{ab} represents a box function over the interval $[\ln a, \ln b]$. Equation (2.51) is not an inverse Fourier transform in the classical sense. Calculating this inverse Fourier transform does not fit into the L^p/L^q theory we have discussed thus far.

Davies and Anderssen introduce a mollified version of B_{ab}, where $\lim_{\epsilon\to 0} B_{ab,\epsilon} = B_{ab}$, defined as:

$$B_{ab,\epsilon}(t) = \frac{1}{\sqrt{2\pi}\epsilon}\int_{-\infty}^{\infty} \exp\left\{-\frac{(t-s)^2}{2\epsilon^2}\right\} B_{ab}(s)\ \mathrm{d}s. \tag{2.52}$$

Calculating the Fourier transform of $B_{ab,\epsilon}$ and then calculating the inverse Fourier transform in equation (2.51), we obtain:

$$\begin{aligned} \Psi_{\epsilon}^{'}(s) &= \mathscr{F}^{-1}\left[\hat{B}_{ab,\epsilon}(p)\cosh\left(\frac{\pi p}{2}\right)\right] \\ &= \frac{1}{\pi}\Re\left[\operatorname{erf}\left(\frac{s-\ln a-(\pi i/2)}{\sqrt{2}\epsilon}\right) - \operatorname{erf}\left(\frac{s-\ln b-(\pi i/2)}{\sqrt{2}\epsilon}\right)\right]. \end{aligned} \tag{2.53}$$

where Davies and Anderssen make use of Fourier transform calculations that they introduce in section two of [20].

Reverting back to original variables, we obtain:

$$\begin{aligned} \varphi_{\epsilon}^{'}(\omega) &= \frac{1}{\pi}\Re\left[\operatorname{erf}\left(\frac{-\ln(a\omega)-(\pi i/2)}{\sqrt{2}\epsilon}\right) - \operatorname{erf}\left(\frac{-\ln(b\omega)-(\pi i/2)}{\sqrt{2}\epsilon}\right)\right] \\ &= \frac{1}{\pi}\Re\left[\operatorname{erf}\left(\frac{\ln(b\omega)+(\pi i/2)}{\sqrt{2}\epsilon}\right) - \operatorname{erf}\left(\frac{\ln(a\omega)+(\pi i/2)}{\sqrt{2}\epsilon}\right)\right]. \end{aligned} \tag{2.54}$$

Next, we perform similar calculations of those above to find a data-functional in terms of the loss modulus:

$$\eta_{ab} = \int_0^{\infty} \phi''(\omega)\, G^{''}(\omega), \tag{2.55}$$

where $\phi''(\omega)$ satisfies (as a result of equation (1.48)):

$$\int_0^\infty \frac{\omega}{1+\omega^2\tau^2}\phi^{''}(\omega)\ \mathrm{d}\omega = \beta_{ab}(\tau)\,. \tag{2.56}$$

Let $\omega\phi^{''}(\omega) = \varphi^{''}(\omega)$ and multiplying both sides of equation (2.56) by τ, we obtain:

$$\int_0^\infty \frac{\tau}{1+\omega^2\tau^2}\varphi^{''}(\omega)\ \mathrm{d}\omega = \tau\beta_{ab}(\tau)\,. \tag{2.57}$$

Making a change of variables: $\omega = e^{-s}$ and $\tau = e^t$ and letting $\varphi^{''}(\omega) = \Psi^{''}(s)$ and $\beta_{ab}(e^t) = B_{ab}(t)$, equation (2.57) becomes:

$$\int_{-\infty}^\infty \operatorname{sech}(s-t)\,\Psi^{''}(s)\ \mathrm{d}s = e^t B_{ab}(t)\,. \tag{2.58}$$

This is the definition of the convolution of sech with $\Psi^{''}$.

That is:

$$\operatorname{sech} * \Psi^{'} = e^t B_{ab}. \tag{2.59}$$

We note again that the following calculations are formal calculations.

Taking Fourier transforms of both sides and assuming that sech, $\hat{\Psi}^{''}$ and $e^t B_{ab}$ are all integrable functions we can write the Fourier transform of a convolution as a product of Fourier transforms, to give:

$$\begin{aligned}
\mathscr{F}\left[\operatorname{sech} * \Psi''\right] &= \mathscr{F}\left[e^t B_{ab}\right] \\
\mathscr{F}\left[\operatorname{sech}\right]\mathscr{F}\left[\Psi''\right] &= \mathscr{F}\left[e^t B_{ab}\right]. \qquad (2.60)
\end{aligned}$$

The Fourier transform of a sech function gives another sech function and, assuming that the Fourier transform of Ψ exists we can write equation (2.59) as:

$$\operatorname{sech}\left(\frac{\pi p}{2}\right)\hat{\Psi}''(p) = \mathscr{F}\left[e^t B_{ab}\right]. \qquad (2.61)$$

Rearranging, we obtain;

$$\begin{aligned}
\hat{\Psi}''(p) &= \mathscr{F}\left[e^t B_{ab}\right]\cosh\left(\frac{\pi p}{2}\right) \\
\Psi''(s) &= \mathscr{F}^{-1}\left[\mathscr{F}\left[e^t B_{ab}\right]\cosh\left(\frac{\pi p}{2}\right)\right]. \qquad (2.62)
\end{aligned}$$

B_{ab} represents a box function over the interval $[\ln a, \ln b]$. We introduce a mollified version of B_{ab}, slightly different to the one introduced previously, where $\lim_{\epsilon\to 0} B_{ab,\epsilon} = B_{ab}$, defined as:

$$B_{ab,\epsilon}(t) = \frac{1}{\sqrt{2\pi}\epsilon}\exp\left\{-t - \frac{1}{2}\epsilon^2\right\}\int_{-\infty}^{\infty}\exp\left\{-\frac{(t-s)^2}{2\epsilon^2}\right\}e^s B_{ab}\left(s-\epsilon^2\right)\,\mathrm{d}s. \qquad (2.63)$$

Calculating the Fourier transform of $e^t B_{ab,\epsilon}$ and then calculating the inverse Fourier transform in equation (2.62), we obtain:

$$\begin{aligned}\Psi_\epsilon^{''}(s) &= \mathscr{F}^{-1}\left[\mathscr{F}\left[e^t B_{ab}\right]\cosh\left(\frac{\pi p}{2}\right)\right] \\ &= \frac{e^s}{\pi}\Re\left[\operatorname{erf}\left(\frac{s-\ln a-(\pi i/2)}{\sqrt{2}\epsilon}\right)-\operatorname{erf}\left(\frac{s-\ln b-(\pi i/2)}{\sqrt{2}\epsilon}\right)\right]. \qquad (2.64)\end{aligned}$$

Reverting back to original variables, we obtain:

$$\varphi_\epsilon^{'}(\omega) = \frac{1}{\pi}\Re\left[\operatorname{erf}\left(\frac{\ln(b\omega)+(\pi i/2)}{\sqrt{2}\epsilon}\right)-\operatorname{erf}\left(\frac{\ln(a\omega)+(\pi i/2)}{\sqrt{2}\epsilon}\right)\right], \qquad (2.65)$$

$$\varphi_\epsilon^{''}(\omega) = -\frac{1}{\pi}\Im\left[\operatorname{erf}\left(\frac{\ln(b\omega)+(\pi i/2)}{\sqrt{2}\epsilon}\right)-\operatorname{erf}\left(\frac{\ln(a\omega)+(\pi i/2)}{\sqrt{2}\epsilon}\right)\right]. \qquad (2.66)$$

We consider calculations for the storage modulus, i.e. we work with $\Psi^{'}(s)$ and $\varphi^{'}(\omega)$. We note that similar calculations can be performed for $\Psi^{''}(s)$ and $\varphi^{''}(\omega)$ derived from the loss modulus.

Now, Davies and Anderssen claim that $\lim_{\epsilon\to 0} B_{ab,\epsilon} = B_{ab}$ implies that $\lim_{\epsilon\to 0}\Psi_\epsilon^{'}(s) = \Psi^{'}(s)$ (or $\lim_{\epsilon\to 0}\varphi_\epsilon^{'}(\omega) = \varphi^{'}(\omega)$ if we revert back to original variables). However, it is not so clear that this is in fact true.

We demonstrate in the calculations that follow that it is true that if $\lim_{\epsilon\to 0} B_{ab,\epsilon} = B_{ab}$ then $\lim_{\epsilon\to 0} \hat{\Psi}_{\epsilon}^{'}(p) = \hat{\Psi}^{'}(p)$ (or $\lim_{\epsilon\to 0} \hat{\varphi}_{\epsilon}^{'}(\omega) = \hat{\varphi}^{'}(\omega)$ if we revert back to original variables).

We have seen from equation (2.51) that the following is true:

$$\hat{\Psi}^{'}(p) = \hat{B}_{ab}(p)\cosh\left(\frac{\pi p}{2}\right) \tag{2.67}$$

and $\hat{\Psi}_{\epsilon}^{'}(p)$ is defined as:

$$\hat{\Psi}_{\epsilon}^{'}(p) = \hat{B}_{ab,\epsilon}(p)\cosh\left(\frac{\pi p}{2}\right). \tag{2.68}$$

Now, $B_{ab,\epsilon}$ is defined, in equation (2.52), as:

$$\begin{aligned} B_{ab,\epsilon}(t) &= \frac{1}{\sqrt{2\pi}\epsilon}\int_{-\infty}^{\infty}\exp\left\{-\frac{(t-s)^2}{2\epsilon^2}\right\}B_{ab}(s)\ \mathrm{d}s \\ &= (f * B_{ab})(t), \end{aligned} \tag{2.69}$$

where

$$f(s) = \frac{1}{\sqrt{2\pi}\epsilon}\exp\left\{-\frac{s^2}{2\epsilon^2}\right\}$$

and $f(s) * B_{ab}(s)$ is defined as the convolution of $f(s)$ with $B_{ab}(s)$. Next we take Fourier transforms of both sides to give:

$$\hat{B}_{ab,\epsilon}(t) = \widehat{f * B_{ab}}. \tag{2.70}$$

Provided that the functions $f(s)$ with $B_{ab}(s)$ are integrable, we can express the Fourier transform of a convolution as the product of two Fourier transforms. (For more detail, see Rudin [71].) We note that $B_{ab}(s)$ is defined as the box function over a finite interval, hence it is integrable. The function $f(s)$ is a Gaussian and hence, is an integrable function.

$$\hat{B}_{ab,\epsilon}(t) = \hat{f}(p)\,\hat{B}_{ab}(p) \tag{2.71}$$

The Fourier transform of $f(s)$ can be calculated without difficulty, to give:

$$\hat{B}_{ab,\epsilon}(t) = \exp\left\{-\epsilon^2 p^2\right\}\hat{B}_{ab}(s). \tag{2.72}$$

Substituting this expression back into equation (2.68)

$$\hat{\Psi}'_{\epsilon}(p) = \exp\left\{-\epsilon^2 p^2\right\}\hat{B}_{ab}(p)\cosh\left(\frac{\pi p}{2}\right). \tag{2.73}$$

Finally, taking limits as ϵ tends to zero, we get:

$$\begin{aligned}\lim_{\epsilon\to 0} \hat{\Psi}'_{\epsilon}(p) &= \lim_{\epsilon\to 0} \exp\left\{-\epsilon^2 p^2\right\} \hat{B}_{ab}(p) \cosh\left(\frac{\pi p}{2}\right) \\ &= \hat{B}_{ab}(p) \cosh\left(\frac{\pi p}{2}\right) \\ &= \hat{\Psi}'(p) . \end{aligned} \tag{2.74}$$

Hence, we have demonstrated that for $\lim_{\epsilon\to 0} B_{ab,\epsilon} = B_{ab}$ it is true that $\lim_{\epsilon\to 0} \hat{\Psi}'_{\epsilon}(p) = \hat{\Psi}'(p)$. The question now is, can we demonstrate that $\lim_{\epsilon\to 0} \Psi'_{\epsilon}(s) = \Psi'(s)$?

Consider the following equation:

$$\begin{aligned}\lim_{\epsilon\to 0} \Psi'_{\epsilon}(s) &= \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \hat{\Psi}'_{\epsilon}(p)\, e^{ips}\, \mathrm{d}p \\ &= \int_{-\infty}^{\infty} \lim_{\epsilon\to 0} \hat{\Psi}'_{\epsilon}(p)\, e^{ips}\, \mathrm{d}p \\ &= \int_{-\infty}^{\infty} \hat{\Psi}'(p)\, e^{ips}\, \mathrm{d}p, \\ &= \Psi'(s) \end{aligned} \tag{2.75}$$

the problem with equation (2.75) as it is, is that it is not obvious that we can take the limit inside the integral. We will demonstrate that the interchange of limits cannot be justified by the Dominated Convergence Theorem. Suppose, for a contradiction, there exists an integrable function $\eta(p)$, such that:

$$\begin{aligned}
\left|\hat{\Psi}_{\epsilon}^{'}(p)\, e^{ips}\right| &= \left|\hat{\Psi}_{\epsilon}^{'}(p)\right| \\
&= \left|\exp\left\{-\epsilon^2 p^2\right\} \hat{B}_{ab}(p) \cosh\left(\frac{\pi p}{2}\right)\right| \leq \eta(p)
\end{aligned} \tag{2.76}$$

for all ϵ and a. e. p. It follows that for a. e. p,

$$\left|\exp\{-1\} \hat{B}_{ab}(p) \cosh\left(\frac{\pi p}{2}\right)\right| \leq \eta(p)\,. \tag{2.77}$$

Now

$$\int_{-\infty}^{\infty} \eta(p)\ \mathrm{d}p \geq \int_{-\infty}^{\infty} \left|\exp\{-1\} \hat{B}_{ab}(p) \cosh\left(\frac{\pi p}{2}\right)\right| \mathrm{d}p = \infty. \tag{2.78}$$

It follows that $\eta(p)$ is not integrable. We deduce that $\left|\hat{\Psi}_{\epsilon}^{'}(p)\, e^{ips}\right|$ does not have a dominator, and hence Davies and Anderssen cannot justify taking the limit into the integral, by the Dominated Convergence Theorem, in equation (2.75). This does not prove that $\lim_{\epsilon\to 0} \Psi_{\epsilon}^{'}(s) \neq \Psi^{'}(s)$, but we can not assume that this is true. Hence, it is not obvious how one would interpret the remaining results of the paper. We will consider this in Chapter 6.

However, if we assume for the moment that $\lim_{\epsilon\to 0} \Psi_{\epsilon}^{'}(s) = \Psi^{'}(s)$, then, Davies and Anderssen proceed in their calculations by considering the expressions that they obtain for both the storage and loss moduli, seen in equations (2.65) and (2.66). They reduce their problem to evaluating expressions of the form:

$$\operatorname{erf}\left(\frac{x + (\pi i/2)}{\sqrt{2}\epsilon}\right). \tag{2.79}$$

Davies and Anderssen obtain their intervals of sampling localisation by taking the limit as $\epsilon \to 0$ of functions that are sums of the above expression. They deduce Theorems 1.1 and 1.2 using asymptotics and results of Abramowitz and Stegun [2] ((7.1.16), page 298). We will discuss the properties of equation (2.79) and hence, the validity of these results later in Chapter 6.

2.6 Summary

In this chapter we have considered the work of Loy, Newbury, Davies and Anderssen [50],[51] and Dodd [23] in great detail. We have worked through the steps in the original calculations of Loy, Newbury, Davies and Anderssen [50] and seen how the ill-posedness of the problem has made the calculation difficult. We have had to introduce a mollifier and take limits in order to be able to evaluate our expression.

We have seen from the work of Renardy [69] that g is an analytic function (which we will demonstrate in Chapter 3) and hence, although the work of Loy, Newbury, Davies and Anderssen [50] and Dodd [23] is correct, it holds for the zero function only. Revised work by Loy, Davies and Anderssen [51] attempts to perform similar calculations without the assumption of g having compact support. There are some difficulties in their approach, which sees them convoluting $g + c$ with f, where c is compactly supported. The result seems to be a consequence of the definition of convolution rather than κ_g being compactly supported in the normal sense. At most, we can say that the bilinear form has compact support in the weak sense.

We have also considered the sampling localisation results of Davies and Anderssen. We have demonstrated that it is impossible to find a dominator to justify writing $\lim_{\epsilon\to 0} \Psi_\epsilon^{'}(s) = \Psi^{'}(s)$, and hence it raises questions on whether or not their calculations are valid. We note that there are other limit theorems that we might consider, namely, the monotone convergence theorem (for details see Theorem 16.1 in Jost [47]). However, to satisfy this theorem, $\hat{\Psi}_\epsilon^{'}(s)$ needs to be a monotonically increasing sequence for all ϵ. Davies and Anderssen note that $\hat{\Psi}_\epsilon^{'}(s)$ consists of an infinite number of pulses as ϵ tends to zero. Clearly $\hat{\Psi}_\epsilon^{'}(s)$ is not monotonically increasing. Hence, demonstrating that $\lim_{\epsilon\to 0} \Psi_\epsilon^{'}(s) = \Psi^{'}(s)$ is true is difficult.

One thing to note, which we will address in great detail in the next chapter, is the Paley-Wiener theorem. In the work of Loy, Newbury, Davies and Anderssen [50], when g was assumed to have compact support, they also demonstrated that they could find an interval of compact support for h using the Paley-Wiener theorem. However, in their revised work, where g is no longer assumed to have compact support, they make no reference to this method. We will address this in the next chapter and prove some interesting results.

Chapter 3

Non-Compactness of Support Using Paley-Wiener in $L^2(\mathbb{R})$

We demonstrated in the previous chapter that with some modifications to their calculations, Loy, Davies and Anderssen [51] are able to obtain an interval of support in a weak sense for the relaxation spectrum for $g \in F_{[\lambda,p]}$, with $1 < p \leq 2$. Now, for $g \in F_{[\lambda,2]}$, there is an alternative method that uses the Paley-Wiener Theorem that can be used to demonstrate that the relaxation spectrum h has compact support. In the original work of Loy, Newbury, Anderssen and Davies [50], (where g is assumed to have compact support) they demonstrate, using the Paley-Wiener Theorem, that the support of the relaxation spectrum h is contained in the same interval as for the case using the bilinear form for all $1 < p \leq 2$.

In the revised paper by Loy, Davies and Anderssen, where g no longer has compact support, they make no reference to the Paley-Wiener Theorem (for $p = 2$). The purpose of this chapter is to address this. We begin by considering modifications of g, and try to satisfy the hypotheses of the Paley-Wiener Theorem. Our only success is achieved by multiplying by a function of compact support, and obtaining an interval of support dependent on the support of the multiplying function, similar to the revised work of Loy, Newbury, Anderssen and Davies [50]. However, we note that we are in fact forcing g to be compactly supported in this case.

The main result of this chapter is a proof demonstrating that in the setting of square integrable relaxation spectra we show that if we can satisfy the conditions of the Paley-Wiener Theorem for h, then g must be trivial. We introduce the following theorem, which we will prove in this chapter:

Theorem 3.1. *For a function g belonging to the space $F_{[\lambda,2]}$, if $\hat{g}$ is entire such that $|\hat{g}(z) \cdot \cosh(\lambda z)| = \left|\hat{h}(z)\right|$ is of exponential type, then $g = 0$ (and $h = 0$).*

This chapter is organised as follows; in section 3.1 we introduce the Paley-Wiener and Plancherel theorems in their L^2 setting and review the work of Loy, Newbury, Anderssen and Davies [50] and Dodd [23]. In section 3.2 we make modifications to our function to demonstrate the difficulty in deducing a bound of exponential type for $|\hat{g}\xi_\lambda|$. We demonstrate Renardy's claim, that $g \in F_{[\lambda,p]}$ means that g is an analytic function and if g is compactly supported then $g = 0$, in section 3.3. In sections 3.4, 3.5 and 3.6 we prove results that lead up to proving Theorem 3.1. The key intermediate step is demonstrating that if $\hat{g}(\cdot)\cosh(\lambda\cdot)$ is of exponential type, then so is $\hat{g}$. A simple argument shows this if g is one-signed; if g is two-signed, we make use of some results from the theory of meromorphic functions.

3.1 Relaxation Spectrum Recovery using the Paley-Wiener Theorem

There is a special case of Conjecture 2.1 for $p = 2$. It makes use of the Plancherel and the Paley-Wiener theorems (see Rudin [71]) which we will introduce now:

Theorem 3.2. *Paley-Wiener Theorem*

Let $f(z)$ be an entire function such that $|f(z)| \leq Ke^{\gamma|z|}$ for some $K \geq 0$ and $\gamma > 0$. If the restriction of f to the real line is in $L^2(\mathbb{R})$, then there exists a function $F(t) \in L^2(-\gamma, \gamma)$ such that

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\gamma}^{\gamma} F(t)e^{izt}\, \mathrm{d}t \tag{3.1}$$

for all z.

An entire function is a function $f : \mathbb{C} \to \mathbb{C}$ which is holomorphic everywhere on the complex domain $\mathbb{C}$. Given a bound of the type in the first line of Theorem 3.2, we say that f is of exponential type.

The Paley-Wiener theorems of the type above give us results of compact support for the complex Fourier transform of a function that has a bound of exponential type. We are familiar with the Fourier transform of a real variable. However, it's not obvious how to interpret the Fourier transform of a complex variable. We refer the reader to section 10.3 of Champeney [17]. Champeney defines the Complex Fourier transform, $F_c(z)$, where $z = y + i\alpha$ as:

$$F_c(z) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i z x}\, \mathrm{d}x = \int_{-\infty}^{\infty} f(x)\, e^{2\pi\alpha x} e^{-2\pi i y x}\, \mathrm{d}x. \tag{3.2}$$

Champeney notes that $F_c(z)$ will be defined at a point $z \in \mathbb{C}$ if and only if $f(x)\, e^{2\pi\alpha x} \in L^1(\mathbb{R})$. If we refer back to equation (3.1), we can see that for the existence of $f(z)$, this must imply certain growth restrictions on $F(t)$. We will need to be aware of this later on as we perform calculations involving the Fourier transform of a complex function.

We introduce another theorem that we will need for the calculations in this chapter:

Theorem 3.3. *Plancherel's Theorem*

One can associate to each $f \in L^2(\mathbb{R})$ a function $\hat{f} \in L^2(\mathbb{R})$ so that the following properties hold:

1. *If $f \in L^1 \bigcap L^2$, then $\hat{f}$ is defined to be the Fourier transform (i.e. via the integral representation) of f;*
2. *For every $f \in L^2$, then $\|\hat{f}\|_2 = \|f\|_2$;*
3. *The mapping $f \to \hat{f}$ is a Hilbert space isomorphism of L^2 onto L^2;*
4. *The following symmetric relation exists between f and $\hat{f}$; If*

$$\varphi_A(t) = \int_{-A}^{A} f(x)\, e^{-ixt}\, \mathrm{d}x \;\; and \;\; \psi_A(x) = \int_{-A}^{A} \hat{f}(t) e^{ixt}\, \mathrm{d}t$$

then

$$\left\|\varphi_A - \hat{f}\right\|_2 \to 0 \;\; and \;\; \|\psi_A - f\|_2 \to 0 \;\; as\, A \to \infty.$$

Now, we review the work of Loy, Newbury, Anderssen and Davies [50] and Dodd [23]. They make use of the above theorems and obtain an interval of support for the relaxation spectrum. Their method makes use of the assumption that the function g as defined in Conjecture 2.1 has compact support, which we now know refers to only the zero function.

Providing we can satisfy that $\hat{g}\xi_\lambda$ is a holomorphic function for all $z \in \mathbb{C}$ (entire) and of exponential type, then we can deduce an interval of support for $\kappa_g(f)$. The integral defined in equation (3.1) is an inverse Fourier transform (with extension to the complex plane). In order to apply the Paley-Wiener Theorem to our problem we need to extend our function $\hat{g} \cdot \xi_\lambda$ to the complex plane. We can easily extend ξ_λ to the complex plane, writing:

$$\xi_\lambda(z) = \frac{e^{-\lambda z} + e^{\lambda z}}{2} \qquad \forall z \in \mathbb{C}. \tag{3.3}$$

If we define:

$$\hat{g}(z) := \int_{-\infty}^{\infty} g(t)\mathrm{e}^{-itz}\,\mathrm{d}t \quad \forall z \in \mathbb{C}, \tag{3.4}$$

then it follows from the Identity Theorem of Complex Analysis, Priestley [68] that equations (3.3) and (3.4) are both unique extensions to the complex plane.

We note that equation (3.4) imposes strong conditions on g. We remind the reader of section 10.3 of Champeney [17] that discusses the complex Fourier transform.

Now, it is obvious that the function $\cosh(\lambda z) = (e^{-\lambda z} + e^{\lambda z})/2$ is holomorphic, but it is not so obvious however that $\hat{g}$ is holomorphic. When g is assumed to have compact support one demonstrates that $\hat{g}$ is continuous, and that:

$$\int_{\partial\Delta} g = 0$$

for every $\partial\Delta$ a closed triangular Jordan curve. It follows from Morera's Theorem that $\hat{g}$ is entire. We refer the reader to Chapter 3 of Dodd [23] for the detailed proof.

We assume for the moment that $\hat{g}$ is an entire function. We wish to find a bound of exponential type for $|\hat{g}(z) \cdot \cosh(\lambda z)|$, that is, we wish to find constants C and A such that:

$$|\xi_\lambda(z) \cdot \hat{g}(z)| \leq C\mathrm{e}^{A|z|}. \tag{3.5}$$

When g was assumed to have compact support, on the interval $[-a, a]$ say, there was no problem in finding a bound of exponential type:

$$\begin{aligned}
|\hat{g}(z)\cosh(\lambda z)| &\leq |\cosh(\lambda z)| \int_{-\infty}^{\infty} \left|g(t)e^{izt}\right| \mathrm{d}t \\
&= \left|\frac{e^{\lambda z} + e^{-\lambda z}}{2}\right| \int_{-a}^{a} \left|g(t)e^{izt}\right| \mathrm{d}t \\
&= \left|\frac{e^{\lambda z} + e^{-\lambda z}}{2}\right| \int_{-a}^{a} |g(t)| \left|e^{\Im(z)t}\right| \mathrm{d}t \\
&\leq e^{\lambda|z|} \sup_{[-a,a]} \left|e^{\Im(z)t}\right| \int_{-a}^{a} |g(t)| \mathrm{d}t \\
&\leq e^{\lambda|z|} e^{a|\Im(z)|} \|g\|_1 \\
&\leq \|g\|_1 e^{(\lambda+a)|z|}.
\end{aligned} \tag{3.6}$$

The Paley-Wiener Theorem yields that $h = \mathscr{F}^{-1}[\hat{g}(r)\cosh(\lambda r)] \in L^2(\mathbb{R})$ and has compact support in $[-\lambda - a, \lambda + a]$.

However, when we take away this assumption on g, this problem becomes very difficult. The methods used in Dodds work [23] rely on g having compact support; the right hand side of the second inequality in equation (3.6) would be infinite if $[-a, a]$ were replaced by $(-\infty, \infty)$. However, the method used by Loy, Davies and Anderssen in their revised paper [51], works for values of p in the interval $1 < p \leq 2$, so if they have proved it holds for $p = 2$ then we should be able to establish a corresponding result via the Paley-Wiener Theorem and Plancherel Theorem for $p = 2$.

3.2 Modification Methods

We ask the question; how could we modify our function $\hat{g}\cosh$ so that we can obtain a bound of exponential type?

In this section we make some modifications to our function $\xi_\lambda \hat{g}$. The first modification we make is to multiply g by a function c_η, with compact support, that tends to 1 pointwise as $\eta \to 0$. We demonstrate how this is similar to the function c that Loy, Davies and Anderssen [51] introduced in their revised paper in the sense that their function c imposed compact support. In our example, since c_η has compact support we can repeat the arguments of inequality (3.6) to find a bound of exponential type. Our second modification is a convolution with a sequence of functions converging to the Dirac mass in the sense of distributions. We illustrate below for a specific choice of c_η.

We define $h_\eta = \mathscr{F}^{-1}\left[\widehat{g \cdot c_\eta}\cosh(\lambda \cdot)\right]$, where c_η is defined as;

$$c_\eta(x) = \begin{cases} 2 - e^{\eta|x|}, & \text{if } |x| < \ln(2)/\eta \\ 0, & \text{if } |x| \geq \ln(2)/\eta. \end{cases} \tag{3.7}$$

We calculate a bound of exponential type:

$$\begin{aligned}
\left|\widehat{g\cdot c_\eta}(z)\cosh(\lambda z)\right| &= \left|\cosh(\lambda z)\right|\left|\int_{-\infty}^{\infty} g(t)\,c_\eta(t)\,e^{-itz}\,\mathrm{d}t\right| \\
&= \left|\frac{e^{\lambda z}+e^{-\lambda z}}{2}\right|\left|\int_{-\ln(2)/\eta}^{\ln(2)/\eta} g(t)\left(2-e^{\eta|t|}\right)e^{-itz}\,\mathrm{d}t\right| \\
&\leq \left|\frac{e^{\lambda|z|}+e^{\lambda|z|}}{2}\right|\int_{-\ln(2)/\eta}^{\ln(2)/\eta}\left|g(t)\left(2-e^{\eta|t|}\right)e^{-itz}\right|\,\mathrm{d}t \\
&\leq e^{\lambda|z|}\int_{-\ln(2)/\eta}^{\ln(2)/\eta}\left|g(t)\,e^{-it(\Re(z)+i\Im(z))}\right|\,\mathrm{d}t \\
&\leq e^{\lambda|z|}\int_{-\ln(2)/\eta}^{\ln(2)/\eta}\left|g(t)\,e^{t\Im(z)}\right|\,\mathrm{d}t \\
&\leq e^{\lambda|z|}e^{|\Im(z)|\ln(2)/\eta}\left\|g\right\|_1 \\
&\leq \left\|g\right\|_1 e^{(\lambda+\ln(2)/\eta)|z|}. \qquad (3.8)
\end{aligned}$$

This is in the correct form to be able to apply the Paley-Wiener Theorem. If we could apply the Paley-Wiener Theorem then h_η would have compact support on the interval $[-\lambda-\ln(2)/\eta, \lambda+\ln(2)/\eta]$. However, since $g\cdot c_\eta \notin F_{[\lambda,2]}$ (unless $g=0$), applying the Paley-Wiener Theorem cannot be justified, and so we must interpret this interval of compact support as support in a weak sense. It does however agree with the interval that Loy, Anderssen and Davies [51] obtained in their revised paper. They obtain an interval of support for $h(=\kappa_g)$ as $[-\lambda-b, \lambda-a]$, where c is supported on $[a,b]$.

We deduce that $\widehat{c_\eta\cdot g}\to\hat{g}$ in $L^2(\mathbb{R})$, by demonstrating that $c_\eta\cdot g\to g$ in $L^2(\mathbb{R})$ (for g belonging to $L^2(\mathbb{R})$) and noting that the Fourier transform is a Hilbert space isomorphism on $L^2(\mathbb{R})$. Consider:

$$\lim_{\eta \to 0} \|(c_\eta \cdot g) - g\|_2^2 = \lim_{\eta \to 0} \int_{-\infty}^{\infty} |(c_\eta(x) - 1)\, g(x)|^2 \,\mathrm{d}x. \tag{3.9}$$

We demonstrate that we can take the limit inside the integral by satisfying the Dominated Convergence Theorem. We refer the reader to Theorem 2.5 in Chapter 2 for details of the theorem. We note that $|(c_\eta(x) - 1)\, g(x)|^2$ is integrable at each η, by definition and that $(c_\eta \cdot g) - g \to 0$ almost everywhere. Finally, we need to demonstrate that $|(c_\eta(x) - 1)\, g(x)|^2 \leq G(x)$ where $G(x)$ is integrable.

For $|x| \geq \ln(2)/\eta$, we have:

$$|(c_\eta(x) - 1)\, g(x)|^2 = |(0 - 1) \cdot g(x)|^2 = |g(x)|^2. \tag{3.10}$$

Furthermore, for $|x| < \ln(2)/\eta$, we have:

$$|(c_\eta(x) - 1)\, g(x)|^2 = \left|\left(1 - e^{\eta|x|}\right) \cdot g(x)\right|^2 \leq |g(x)|^2, \tag{3.11}$$

where $|g(x)|^2$ is integrable by definition. We have a dominator and hence it follows from the Dominated Convergence Theorem that we can take the limit into the integral in equation (3.9), allowing us to write:

$$\begin{aligned}
\lim_{\eta\to 0}\int_{-\infty}^{\infty}\left|\left(c_\eta(x)-1\right)\cdot g(x)\right|^2\,\mathrm{d}x &= \lim_{\eta\to 0}\int_{-\ln(2)/\eta}^{\ln(2)/\eta}\left|\left(2-e^{\eta|x|}-1\right)\cdot g(x)\right|^2\,\mathrm{d}x \\
&= \int_{-\ln(2)/\eta}^{\ln(2)/\eta}\lim_{\eta\to 0}\left|\left(1-e^{\eta|x|}\right)\cdot g(x)\right|^2\,\mathrm{d}x \\
&= \int_{-\ln(2)/\eta}^{\ln(2)/\eta}\left|\left(1-e^{0}\right)\cdot g(x)\right|^2\,\mathrm{d}x \\
&= 0. \qquad (3.12)
\end{aligned}$$

It follows that $c_\eta\cdot g\to g$ in $L^2(\mathbb{R})$ (for g belonging to $L^2(\mathbb{R})$) and since the Fourier transform is a Hilbert space isomorphism on $L^2(\mathbb{R})$, we deduce from the Plancherel Theorem (Theorem 3.3) that $\widehat{c_\eta\cdot g}\to\hat{g}$ in $L^2(\mathbb{R})$. We remind the reader however, that $\widehat{c_\eta\cdot g}\cdot\xi_\lambda\notin L^2(\mathbb{R})$ (unless $g=0$).

If we could apply the Paley-Wiener Theorem, the support of h_η would be contained in $[-\lambda-\ln(2)/\eta,\lambda+\ln(2)/\eta]$ and the length of this interval tends to infinity as $\eta\to 0$. This result does not really tell us very much about g or h though, since we have imposed compact support on $g\cdot c_\eta$, so $g\cdot c_\eta\notin F_{[\lambda,2]}$. Therefore finding a bound of exponential type for $|\widehat{g\cdot c_\eta}\cdot\cosh|$ is not addressing our problem. Ideally we would like a result where we are not performing calculations on functions outside of our space $F_{[\lambda,2]}$.

Instead of working with $\hat{g}\cdot\cosh$, we consider $\widehat{g*b_\eta}\cdot\cosh$, where $b_\eta(x)$ is defined as follows for $\eta>0$,

$$b_\eta(x)=\begin{cases}\frac{e^{-x^2/\eta^2}}{\sqrt{\pi}\eta}, & \text{if } |x|<1/\eta\\ 0, & \text{if } |x|\geq 1/\eta,\end{cases} \qquad (3.13)$$

where $\eta > 0$. b_η has the following graph:

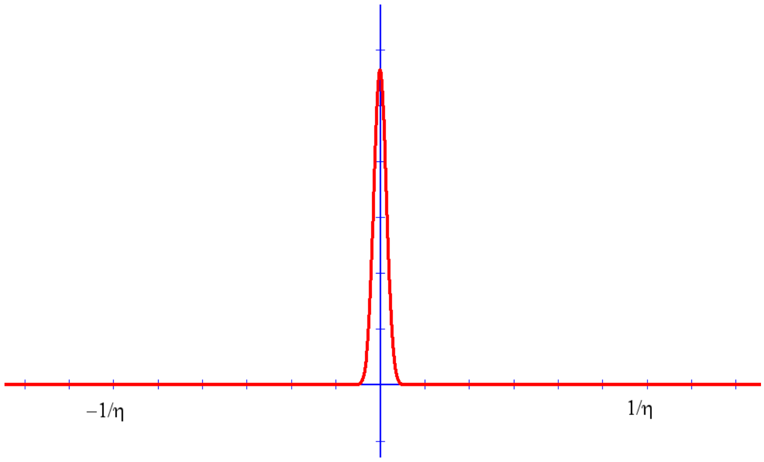

Figure 3.1: graph of b_η for small η

We note that the function above is not continuous, it does have small jumps at $x = \pm 1/\eta$.

Now, we wish to show that as $\eta \to 0$ the function $b_\eta \to \delta$ in the sense of distributions, where δ is the Dirac mass.

Consider:

$$\tilde{I} = \int_{-\infty}^{\infty} \varphi(x)\, b_\eta(x)\ \mathrm{d}x, \tag{3.14}$$

where φ is a test function for which φ and all of its derivatives of all orders (which are continuous) vanish outside some compact subset of $\mathbb{R}$. Let φ and φ' be supported on the interval $(-k, l)$. Then, for sufficiently large η we have:

$$\begin{aligned}\tilde{I} &= \int_{-1/\eta}^{1/\eta} \varphi(x) \frac{e^{-x^2/\eta^2}}{\eta} \, \mathrm{d}x \\ &= \int_{-k}^{0} \varphi(x) \frac{e^{-x^2/\eta^2}}{\eta} \, \mathrm{d}x + \int_{0}^{l} \varphi(x) \frac{e^{-x^2/\eta^2}}{\eta} \, \mathrm{d}x. \end{aligned} \tag{3.15}$$

We use integration by parts to give:

$$\begin{aligned}\tilde{I} &= \left[\varphi(x) \frac{\sqrt{\pi}\eta}{2\sqrt{\pi}\eta} \operatorname{erf}(x/\eta)\right]_{-k}^{0} - \int_{-k}^{0} \varphi^{'}(x) \frac{\sqrt{\pi}\eta}{2\sqrt{\pi}\eta} \operatorname{erf}(x/\eta) \, \mathrm{d}x \\ &+ \left[\varphi(x) \frac{\sqrt{\pi}\eta}{2\sqrt{\pi}\eta} \operatorname{erf}(x/\eta)\right]_{0}^{l} - \int_{0}^{l} \varphi^{'}(x) \frac{\sqrt{\pi}\eta}{2\sqrt{\pi}\eta} \operatorname{erf}(x/\eta) \, \mathrm{d}x, \end{aligned} \tag{3.16}$$

where the antiderivative of $\exp\{-x^2/\eta^2\} = (\eta\sqrt{\pi}/2) \operatorname{erf}(x/\eta)$.

Now, noting that φ vanishes outside the interval $(-k, l)$, we can write equation (3.16) as:

$$\begin{aligned}\tilde{I} &= \varphi(0) \frac{\operatorname{erf}(0)}{2} - \frac{1}{2} \int_{-k}^{0} \varphi^{'}(x) \operatorname{erf}(x/\eta) \, \mathrm{d}x \\ &- \varphi(0) \frac{\operatorname{erf}(0)}{2} - \frac{1}{2} \int_{0}^{l} \varphi^{'}(x) \operatorname{erf}(x/\eta) \, \mathrm{d}x \\ &= -\frac{1}{2} \int_{-k}^{0} \varphi^{'}(x) \operatorname{erf}(x/\eta) \, \mathrm{d}x - \frac{1}{2} \int_{0}^{l} \varphi^{'}(x) \operatorname{erf}(x/\eta) \, \mathrm{d}x. \end{aligned} \tag{3.17}$$

Next, we define the following function:

$$\chi_{[-1/\eta,1/\eta]}(x) = \begin{cases} 1, & \text{if } |x| \leq 1/\eta \\ 0, & \text{otherwise.} \end{cases}$$

Then we can express equation (3.17) as:

$$\tilde{I} = -\frac{1}{2}\int_{\mathbb{R}} \chi_{[-1/\eta,1/\eta]}(x)\,\varphi^{'}(x)\,\mathrm{erf}(x/\eta)\,\mathrm{d}x. \tag{3.18}$$

Noting that $\mathrm{erf}(x/\eta) \in L^1_{\mathrm{loc}}$, then, taking limits as $\eta \to 0$:

$$\begin{aligned} \lim_{\eta\to 0}\tilde{I} &= -\lim_{\eta\to 0}\int_{\mathbb{R}} \varphi(x)\,b_\eta(x)\,\mathrm{d}x \\ &= -\frac{1}{2}\int_{\mathbb{R}} \lim_{\eta\to 0}\chi_{[-1/\eta,1/\eta]}(x)\,\varphi^{'}(x)\,\mathrm{erf}(x/\eta)\,\mathrm{d}x \qquad (3.19) \\ &= \frac{1}{2}\int_{-k}^{0}\varphi^{'}(x)\,\mathrm{d}x - \frac{1}{2}\int_{0}^{l}\varphi^{'}(x)\,\mathrm{d}x \\ &= \frac{1}{2}\varphi(0) + \frac{1}{2}\varphi(0) = \varphi(0). \qquad (3.20) \end{aligned}$$

Where we have made use of the Dominated Convergence Theorem in equation (3.18) (see Theorem 2.5) in taking the limit inside the integral.

By way of explanation we note that:

$$\left|\chi_{[-1/\eta,1/\eta]}(x)\,\varphi^{'}(x)\,\mathrm{erf}(x/\eta)\right| \leq \left|\varphi^{'}(x)\right| \leq M. \tag{3.21}$$

We deduce this from the fact that $\varphi^{'}$ is a continuous function with compact support, and the limit as $\eta \to 0$ of $\operatorname{erf}(x/\eta) = 1$ for a.e. $x \in (0, \infty)$, and the limit as $\eta \to 0$ of $\operatorname{erf}(x/\eta) = -1$ for a.e. $x \in (-\infty, 0)$.

We have shown that;

$$\lim_{\eta \to 0} \int_{\mathbb{R}} \varphi(x)\, b_\eta(x)\, \mathrm{d}x = \varphi(0), \tag{3.22}$$

from which we can deduce that as $\eta \to 0$, $b_\eta \to \delta$ in the sense of distributions.

It follows that:

$$\lim_{\eta \to 0} \widehat{g * b_\eta} = \lim_{\eta \to 0} \hat{g} \cdot \hat{b_\eta} = \hat{g} \cdot \hat{\delta} = \hat{g} \cdot 1 = \hat{g}, \tag{3.23}$$

in the sense of distributions. We have made use of a result in Rudin [71], which allows us to express the Fourier transform of a convolution as the product of two Fourier transforms, providing both functions are integrable. The function g must be such that $\hat{g}(z) \cosh(\lambda z) \in L^2$, hence, g is integrable from results of Champeney [17] that we discuss in section 3.1. Furthermore, we can see from the definition of b_η that it is integrable, hence the step above is justified.

To study the support of $h = \mathscr{F}^{-1}[\hat{g}\xi_\lambda]$, in the light of equation (3.23) we seek a bound of exponential type for $\widehat{g * b_\eta}\xi_\lambda$.

We want a bound of exponential type for $\xi_\lambda \cdot \widehat{g * b_\eta}$.

Now:

$$\begin{aligned}
\left|\widehat{g * b_\eta}\right|(z) &= \left|\int_{-\infty}^{\infty} (g * b_\eta)(x)\, e^{-izx}\,\mathrm{d}x\right| \\
&= \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y)\, b_\eta(x-y)\,\mathrm{d}y\, e^{-izx}\,\mathrm{d}x\right| \\
&= \left|\int_{-\infty}^{\infty}\int_{-1/\eta}^{1/\eta} g(y)\,\frac{e^{-(x-y)^2/\eta^2}}{\eta}\,\mathrm{d}y\, e^{-izx}\,\mathrm{d}x\right| \\
&= \left|\int_{-\infty}^{\infty}\frac{1}{\eta}\int_{-1/\eta}^{1/\eta} g(y)\exp\left\{-\frac{1}{\eta^2}\left(x^2-2yx+y^2\right)\right\}\mathrm{d}y\, e^{-izx}\,\mathrm{d}x\right|. \quad (3.24)
\end{aligned}$$

Assuming that we can change the order of integration, then:

$$\begin{aligned}
\left|\widehat{g * b_\eta}\right|(z) &= \left|\frac{1}{\eta}\int_{-1/\eta}^{1/\eta} g(y)\, e^{-y^2/\eta^2}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{\eta^2}\left(x^2-2yx\right)\right\} e^{-izx}\,\mathrm{d}x\,\mathrm{d}y\right| \\
&= \left|\frac{1}{\eta}\int_{-1/\eta}^{1/\eta} g(y)\, e^{-y^2/\eta^2}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{\eta^2}\left(x^2-2yx+i\eta^2 zx\right)\right\}\mathrm{d}x\,\mathrm{d}y\right| \\
&= \left|\frac{1}{\eta}\int_{-1/\eta}^{1/\eta} g(y)\, e^{-y^2/\eta^2}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{\eta^2}\left[x^2+Ax\right]\right\}\mathrm{d}x\,\mathrm{d}y\right| \\
&= \left|\frac{1}{\eta}\int_{-1/\eta}^{1/\eta} g(y)\, e^{-y^2/\eta^2}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{\eta^2}\left[x+\frac{A}{2}\right]^2+\frac{A^2}{4\eta^2}\right\}\mathrm{d}x\,\mathrm{d}y\right|, \quad (3.25)
\end{aligned}$$

where $A = i\eta^2 z - 2y$. Writing $z = u + iv$, and using a substitution $t = x - y - \eta^2 v/2$ we can express equation (3.25) as:

$$\left|\widehat{g * b_\eta}\right| = \left|\frac{1}{\eta}\int_{-1/\eta}^{1/\eta} g(y)\, e^{-y^2/\eta^2} e^{A^2/4\eta^2}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{\eta^2}\left[t+\frac{i\eta^2 u}{2}\right]^2\right\}\mathrm{d}t\,\mathrm{d}y\right| \quad (3.26)$$

We note a result from Papoulis [66], where he demonstrates calculations for the Fourier

Transform of a complex Gaussian. He deduces that for $p \in \mathbb{C}$, we have the following:

$$\int_{-\infty}^{\infty} \exp\left\{-p\left(t+\frac{iw}{2p}\right)^2\right\} \mathrm{d}t = \sqrt{\frac{\pi}{p}} \tag{3.27}$$

Comparing this with equation (3.26), we can deduce that

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{\eta^2}\left[t+\frac{i\eta^2 u}{2}\right]^2\right\} \mathrm{d}t = \eta\sqrt{\pi}. \tag{3.28}$$

Substituting this expression back into equation (3.26), we have:

$$\begin{aligned}
\left|\widehat{g * b_\eta}\right| &= \left|\frac{1}{\eta}\int_{-1/\eta}^{1/\eta} g(y)\, e^{-y^2/\eta^2} \exp\left\{\frac{A^2}{4\eta^2}\right\} \eta\sqrt{\pi}\, \mathrm{d}y\right| \\
&= \left|\sqrt{\pi}\int_{-1/\eta}^{1/\eta} g(y)\, e^{-y^2/\eta^2} \exp\left\{\frac{A^2}{4\eta^2}\right\} \mathrm{d}y\right| \\
&= \left|\sqrt{\pi}\int_{-1/\eta}^{1/\eta} g(y) \exp\left\{-\frac{y^2}{\eta^2}+\frac{\left(i\eta^2 z-2y\right)^2}{4\eta^2}\right\} \mathrm{d}y\right| \\
&= \left|\sqrt{\pi}\int_{-1/\eta}^{1/\eta} g(y) \exp\left\{\frac{-\eta^2 z^2}{4}\right\} \exp\{-iyz\}\, \mathrm{d}y\right|. \qquad (3.29)
\end{aligned}$$

Noting that we can write $z = \Re(z) + i\Im(z)$ and that $\left|e^{ia}\right| = 1$ for a a real number, we can estimate equation (3.29) as:

$$\begin{aligned}
\left|\widehat{g * b_\eta}\right|(z) &\leq \sqrt{\pi}\int_{-1/\eta}^{1/\eta} \left|g(y) \exp\left\{\frac{-\eta^2\left(\Re(z)+i\Im(z)\right)^2}{4}\right\} \exp\{-iy\left(\Re(z)+i\Im(z)\right)\}\right| \mathrm{d}y \\
&\leq \sqrt{\pi}\int_{-1/\eta}^{1/\eta} \left|g(y) \exp\left\{\frac{-\eta^2\left(\Re(z)\right)^2+\eta^2\left(\Im(z)\right)^2}{4}\right\} \exp\{y\Im(z)\}\right| dy. \qquad (3.30)
\end{aligned}$$

Furthermore, we know that $\left|e^{-\eta^2(\Re(z))^2}\right| \leq 1$, hence:

$$\begin{aligned}\left|\widehat{g * b_\eta}\right|(z) &\leq \sqrt{\pi}\int_{-1/\eta}^{1/\eta}\left|g(y)\exp\left\{\frac{\eta^2(\Im(z))^2}{4}\right\}\exp\{y\Im(z)\}\right|\,\mathrm{d}y \\ &\leq \sqrt{\pi}\exp\left\{\frac{\eta^2(\Im(z))^2}{4}\right\}\exp\left\{\frac{|\Im(z)|}{\eta}\right\}\int_{-1/\eta}^{1/\eta}|g(y)|\,\mathrm{d}y. \end{aligned} \tag{3.31}$$

Finally, making use of the fact that $g \in L^1(\mathbb{R})$ we have:

$$\begin{aligned}\left|\widehat{g * b_\eta}\right|(z) &\leq \sqrt{\pi}\,\|g\|_1\exp\left\{\frac{\eta^2(\Im(z))^2}{4}\right\}\exp\left\{\frac{|\Im(z)|}{\eta}\right\} \\ &\leq \sqrt{\pi}\,\|g\|_1\exp\left\{\frac{\eta^2|z|^2}{4}\right\}\exp\left\{\frac{|z|}{\eta}\right\}. \end{aligned} \tag{3.32}$$

We are seeking a bound of exponential type for $\xi_\lambda \cdot \widehat{g * b_\eta}$. Now:

$$\left|\xi_\lambda \cdot \widehat{g * b_\eta}\right|(z) \leq e^{\lambda|z|}\sqrt{\pi}\,\|g\|_1\exp\left\{\frac{\eta^2|z|^2}{4}\right\}\exp\left\{\frac{|z|}{\eta}\right\}. \tag{3.33}$$

This is not in the form we need to be able to apply the Paley-Wiener Theorem.

The calculations above have demonstrated the difficulty in finding a bound of exponential type for $|\hat{g}(z)\cosh(\lambda z)|$ and hence an interval of compact support for the relaxation spectrum. It is clear that there are elements of $F_{[\lambda,2]}$ such that $h = \mathscr{F}^{-1}[\xi_\lambda \cdot \hat{g}]$ does not have compact support. Calculations using a Gaussian function will clearly demonstrate this. But are there any non-trivial g for which h has compact support? In the following sections we will prove that the answer to this question is no.

3.3 Analyticity

In this section we demonstrate that $g \in F_{[\lambda,p]}$ is an analytic function and furthermore we show that an analytic function with compact support must be the zero function. We demonstrate the following proposition:

Proposition 3.4. *For a function g belonging to the space $F_{[\lambda,p]}$, defined in equation (2.2), g is an analytic function. Furthermore, if g has compact support then g must be the zero function.*

Proof of Proposition 3.4

We demonstrate, for g belonging to $F_{[\lambda,p]}$, that g is bounded:

We make use of arguments presented in Loy et. al. [51].

Let $r, s, t \in \mathbb{R}$ and $|s| < \lambda$,

$$|\exp(ir(t+is))| = |\exp(irt - rs)| = |\exp(irt)|\,|\exp(-rs)| = |\exp(-rs)|\,. \quad (3.34)$$

We can write

$$\begin{aligned} |\exp(ir(t+is))| &= |\exp(-rs)| = 2\frac{|\exp(-rs)|}{(e^{\lambda r} + e^{-\lambda r})}\cosh(\lambda r) \\ &\leq 2\frac{\exp(|rs|)}{e^{\lambda|r|}}\cosh(\lambda r) = 2\exp(|rs| - \lambda|r|)\,\xi_\lambda(r) \\ &= 2\exp(|r|(|s| - \lambda))\,\xi_\lambda(r)\,. \end{aligned} \quad (3.35)$$

It follows that:

$$\begin{aligned} |g(z)| &= |g(t+is)| \\ &\leq \int_{-\infty}^{\infty} |\hat{g}(r)\exp(ir(t+is))| \,\mathrm{d}r \\ &\leq 2\int_{-\infty}^{\infty} e^{|r|(|s|-\lambda)} |\xi_\lambda(r)\,\hat{g}(r)| \,\mathrm{d}r. \end{aligned} \tag{3.36}$$

Equation (3.36) is finite for $g \in F_{[\lambda,p]}$ by Hölders inequality, since $r \to e^{|r|(|s|-\lambda)}$ lies in $L^q(\mathbb{R})$, and $\xi_\lambda(r)\,\hat{g}(r) \in L^p(\mathbb{R})$ by definition. It follows that g is bounded.

Let $z = t + is$, then

$$g(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(r)\, e^{irz}\,\mathrm{d}r. \tag{3.37}$$

We note that every function which is holomorphic in an open set G is analytic in that set (for more detail see Priestley [68]).

We can write $z = \Re(z) + i\Im(z)$.

Demonstrating that $g(z) = u(z) + iv(z)$ is holomorphic is equivalent to showing that u and v exist, are continuous and satisfy:

$$\frac{\partial u}{\partial \Re(z)} = \frac{\partial v}{\partial \Im(z)} \text{ and } \frac{\partial u}{\partial \Im(z)} = -\frac{\partial v}{\partial \Re(z)}; \tag{3.38}$$

the Cauchy-Riemann equations. We split $\hat{g}$ into its real and imaginary parts; $\hat{g}_{\Re}$ and $\hat{g}_{\Im}$ respectively.

We rewrite equation (3.38) as the following:

$$\begin{aligned} g(z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\hat{g}_{\Re}(r) + i\hat{g}_{\Im}(r)\right)(r)\, e^{ir(\Re(z)+i\Im(z))}\,\mathrm{d}r \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\hat{g}_{\Re}(r) + i\hat{g}_{\Im}(r)\right)(r)\, e^{-r\Im(z)}e^{ir\Re(z)}\,\mathrm{d}r \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\hat{g}_{\Re}(r) + i\hat{g}_{\Im}(r)\right) e^{-r\Im(z)}\left[\cos\left(r\Re(z)\right) + i\sin\left(r\Re(z)\right)\right]\,\mathrm{d}r \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-r\Im(z)}\left[\hat{g}_{\Re}(r)\cos\left(r\Re(z)\right) - \hat{g}_{\Im}(r)\sin\left(r\Re(z)\right)\right]\,\mathrm{d}r \\ &+ i\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-r\Im(z)}\left[\hat{g}_{\Re}(r)\sin\left(r\Re(z)\right) + \hat{g}_{\Im}(r)\cos\left(r\Re(z)\right)\right]\,\mathrm{d}r. \end{aligned} \tag{3.39}$$

Where;

$$u(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-r\Im(z)}\left[\hat{g}_{\Re}(r)\cos\left(r\Re(z)\right) - \hat{g}_{\Im}(r)\sin\left(r\Re(z)\right)\right]\,\mathrm{d}r, \tag{3.40}$$

and

$$v(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-r\Im(z)}\left[\hat{g}_{\Re}(r)\sin\left(r\Re(z)\right) + \hat{g}_{\Im}(r)\cos\left(r\Re(z)\right)\right]\,\mathrm{d}r. \tag{3.41}$$

Since $u(z)$ and $v(z)$ are integral equations, calculating the Cauchy-Riemann equations will require taking the derivative into the integral. We demonstrate below how one might be able to justify taking the derivative into the integral. Consider the following equation:

$$\Phi(x) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y. \tag{3.42}$$

Suppose we wish to take the derivative of $\Phi(x)$ with respect to x, that is, we would like to take the derivative inside the integral. We can define the derivative from first principles as:

$$\begin{aligned}\frac{\mathrm{d}\Phi}{\mathrm{d}x} &= \lim_{h\to 0}\frac{\Phi(x+h)-\Phi(x)}{h} \\ &= \lim_{h\to 0}\int_{-\infty}^{\infty}\frac{f(x+h,y)-f(x,y)}{h}\,\mathrm{d}y. \end{aligned} \tag{3.43}$$

We can take the limit inside the integral providing we can satisfy the Dominated Convergence theorem, that is, we can find a dominator for the integrand. Assuming for the moment that we can, then equation (3.43) becomes:

$$\begin{aligned}\frac{\mathrm{d}\Phi}{\mathrm{d}x} &= \int_{-\infty}^{\infty}\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\frac{\partial f(x,y)}{\partial x}\,\mathrm{d}y. \end{aligned} \tag{3.44}$$

It follows that we can take the derivative inside the integral in equations (3.40) and

(3.41) providing we have dominators for the following two integrands:

$$\begin{aligned}
|I_u| &= \frac{e^{-(r+h)\Im(z)}}{h} \left|\hat{g}_{\Re}(r+h)\cos((r+h)\Re(z)) - \hat{g}_{\Im}(r+h)\sin((r+h)\Re(z))\right. \\
&- \left.\hat{g}_{\Re}(r)\cos(r\Re(z)) + \hat{g}_{\Im}(r)\sin(r\Re(z))\right|, \qquad (3.45)
\end{aligned}$$

and

$$\begin{aligned}
|I_v| &= \frac{e^{-(r+h)\Im(z)}}{h} \left|\hat{g}_{\Re}(r+h)\sin((r+h)\Re(z)) + \hat{g}_{\Im}(r+h)\cos((r+h)\Re(z))\right. \\
&- \left.\hat{g}_{\Re}(r)\sin(r\Re(z)) - \hat{g}_{\Im}(r)\cos(r\Re(z))\right|. \qquad (3.46)
\end{aligned}$$

We note that the result, that $\hat{g}(r)\cosh(\lambda r) \in L^p(\mathbb{R})$ implies that g is analytic, is stated in Renardy [69] and partly proven in the work of Loy, Davies and Anderssen [51]. Hence we will assume that we can find dominators such that we can take the derivatives inside the integrals. The following calculations are formal.

$$\begin{aligned}
\frac{\partial u}{\partial \Re(z)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-r\Im(z)} \left[\hat{g}_{\Re}(r)\frac{\partial \cos(r\Re(z))}{\partial \Re(z)} - \hat{g}_{\Im}(r)\frac{\partial \sin(r\Re(z))}{\partial \Re(z)}\right] \mathrm{d}r \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} (-r)\, e^{-r\Im(z)} \left[\hat{g}_{\Re}(r)\sin(r\Re(z)) + \hat{g}_{\Im}(r)\cos(r\Re(z))\right] \mathrm{d}r, \qquad (3.47)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial v}{\partial \Im(z)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial e^{-r\Im(z)}}{\partial \Im(z)} \left[\hat{g}_{\Re}(r)\sin(r\Re(z)) + \hat{g}_{\Im}(r)\cos(r\Re(z))\right] \mathrm{d}r \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} (-r)\, e^{-r\Im(z)} \left[\hat{g}_{\Re}(r)\sin(r\Re(z)) + \hat{g}_{\Im}(r)\cos(r\Re(z))\right] \mathrm{d}r. \qquad (3.48)
\end{aligned}$$

Clearly equation (3.47) is equal to equation (3.48); that is we have satisfied the first of the Cauchy-Riemann equations.

For the second Cauchy-Riemann equation;

$$\begin{aligned}\frac{\partial u}{\partial \Im(z)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial e^{-r\Im(z)}}{\partial \Im(z)} \left[\hat{g}_{\Re}(r)\cos(r\Re(z)) - \hat{g}_{\Im}(r)\sin(r\Re(z))\right] \mathrm{d}r \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (-r)\, e^{-r\Im(z)} \left[\hat{g}_{\Re}(r)\cos(r\Re(z)) - \hat{g}_{\Im}(r)\sin(r\Re(z))\right] \mathrm{d}r, \qquad (3.49)\end{aligned}$$

$$\begin{aligned}\frac{\partial v}{\partial \Re(z)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-r\Im(z)} \left[\hat{g}_{\Re}(r)\frac{\partial \sin(r\Re(z))}{\partial \Re(z)} + \hat{g}_{\Im}(r)\frac{\partial \cos(r\Re(z))}{\partial \Re(z)}\right] \mathrm{d}r \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} re^{-r\Im(z)} \left[\hat{g}_{\Re}(r)\cos(r\Re(z)) - \hat{g}_{\Im}(r)\sin(r\Re(z))\right] \mathrm{d}r. \qquad (3.50)\end{aligned}$$

Once again we can see that equation (3.49) is equal to the negative of equation (3.50).

We have satisfied the Cauchy-Riemann equations for the function $g(z)$. This is in agreement with the results of Renardy [69] and Loy, Davies and Anderssen [51], i.e. the function $g(z)$ is holomorphic, which is equivalent to analytic in the complex plane.
We have demonstrated that $g(z)$ is infinitely differentiable for all z, hence $g(z)$ is an analytic function on the strip $\{z = t + is : -\infty < t < \infty, |s| < \lambda\}$ about the x-axis.

Finally, we note that if $g : \mathbb{R} \to \mathbb{R}$ has compact support, then the set of zeros of g has a limit point on the strip. The Identity Theorem (see Priestley [68]) yields that $g = 0$ on the strip; restricted to the real line, $g = 0$.

We conclude that if $g \in F_{[\lambda,p]}$ and has compact support then g must be the zero function.

3.3.1 Fourier Transforms of Functions with Compact Support

In addition to the result above, there are results in the literature that suggest that for our problem, as it is defined, g cannot be compactly supported. Furthermore, many possible functions exist for $\hat{g}$ that result in h, the relaxation spectrum, not being compactly supported.

In a short, but clear paper Ingham [44] gives us an important result. Ingham poses the following question:

How small can the Fourier transform

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{iyx} \tag{3.51}$$

of $f(x)$ be for large $|y|$, if $f(x)$ has compact support (and is not null)?

Ingham states that $\hat{f}(y)$ cannot satisfy a condition of the type:

$$\hat{f}(y) = O\left(e^{-|y|\epsilon}\right),\ y \to \pm\infty \tag{3.52}$$

where ϵ is a fixed positive number.

We know that for $\hat{g}\xi_\lambda$ to belong to L^p, then $\hat{g}$ must have far field exponential decay, hence it follows from the result of Ingham [44] that g cannot have compact support except for the trivial case.

For further details on results of this form, the reader is directed to Hormander [42], Beurling and Malliavin [12] and Rudin [73].

A well known result, which we can see in Chapter 7 of Strichartz [76], is that the Fourier transform of a continuous function with compact support is analytic. As we have seen from the calculations above, an analytic function cannot have compact support unless it is zero.

We can use this result to deduce that there are certainly functions in $F_{[\lambda,p]}$ such that the relaxation spectrum cannot have compact support. For example, any function $\hat{g}$ that is compactly supported will result in $\hat{g} \cdot \xi_\lambda \in L^p$ (g will not be compactly supported from the result above). The function $\hat{g} \cdot \xi_\lambda$ will also be compactly supported and hence, it follows from the result of Strichartz [76] that h will be an analytic function and cannot have compact support.

We conclude from this section that for g belonging to $F_{[\lambda,p]}$, g is not compactly supported unless $g = 0$. Furthermore, we have also demonstrated that for certain functions in $F_{[\lambda,p]}$, such that $\hat{g}$ is compactly supported, h, the relaxation spectrum cannot have compact support.

These results will be used in the following sections to prove results regarding the support of the relaxation spectrum h.

3.4 Non-compact Support for Non-trivial Functions

At the beginning of this chapter we demonstrated the difficulty in obtaining a bound of exponential type for $|\hat{g}(z) \cdot \cosh(\lambda z)|$ when g is not compactly supported. We also noted the difficulty in showing that $\hat{g}$ is an entire function when g is not compactly supported.

In this section we will assume that $\hat{g}$ is an entire function and in addition to this, that we can find a bound of exponential type for $|\hat{g}(z) \cdot \cosh(\lambda z)|$. By making these assumptions we can show that the corresponding h defined by $h = \mathscr{F}^{-1}[\hat{g}\xi_\lambda]$ has to be trivial (see Theorem 3.1).

The key intermediate result is demonstrating that if $\hat{g}\xi_\lambda$ is of exponential type, then so is $\hat{g}$. For one-signed g we can argue directly. We begin by introducing the following lemma:

Lemma 3.5. *For $g \in F_{[\lambda,2]}$ and either $g \geq 0$ or $g \leq 0$ (g one signed), if $|\hat{g}(z) \cdot \cosh(\lambda z)|$ is of exponential type then $|\hat{g}(z)|$ is also of exponential type.*

We refer the reader to a result in section 10.3 of Champeney [17], that we referred to at the beginning of section 3.1, that discussed the Fourier transform of a complex function, that puts certain constraints on $g(t)$ in the proof below.

Proof of Lemma 3.5

Suppose that:

$$|\hat{g}(z) \cdot \cosh(\lambda z)| \leq Ce^{A|z|} \tag{3.53}$$

for constants $C \geq 0$ and $A > 0$. That is, $|\hat{g}(z) \cdot \cosh(\lambda z)|$ has a bound of exponential type. We note also that $\lambda = \pi/2$.

Initially we restrict our attention to the imaginary axis. We split the imaginary axis into two regions in considering equation (3.53). We consider 2 cases, (See Figure 3.2):

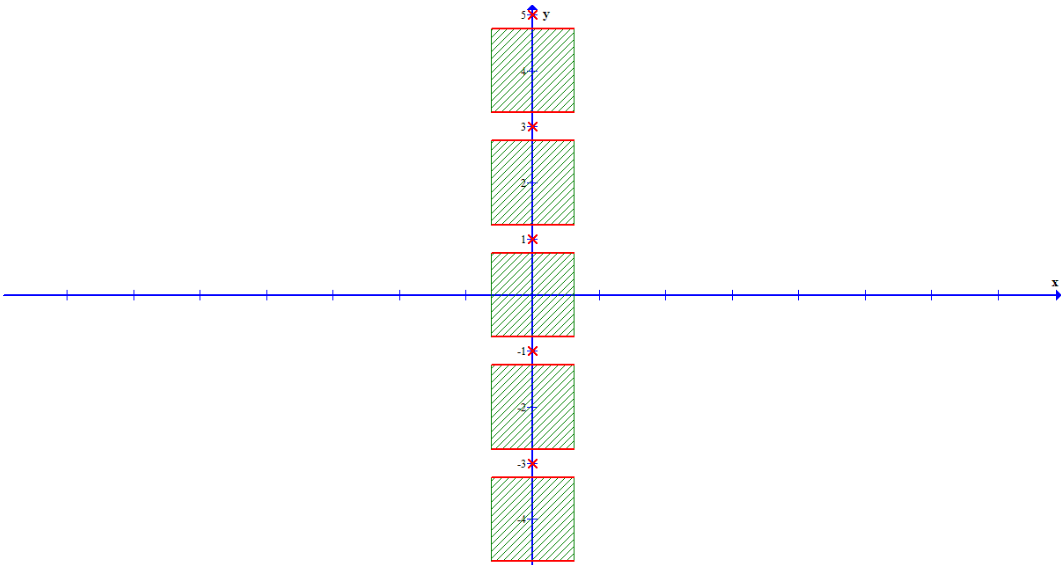

Figure 3.2: Complex Plane

1. Case 1: refers to the shaded areas around the even integers on the imaginary axis. We let the width of these shaded rectangles be 3/2, a distance of 3/4 above and below each even integer.

2. Case 2: refers to the areas around the odd integers on the imaginary axis. The

width of these are 1/2, a distance of 1/4 above and below the odd integers. The zeros of $\cosh(\lambda z)$ are taken at $z = i(2k+1)$ for $k \in \mathbb{Z}$.

Case 1:

We consider case 1, where we are dealing with z in the shaded rectangles containing the even integers.

We note that:

$$|\hat{g}(z)| \leq \frac{Ce^{A|z|}}{|\cosh(\lambda z)|} = \frac{2Ce^{A|z|}}{|e^{\lambda z} + e^{-\lambda z}|}. \tag{3.54}$$

We note here that we must use two different approaches for case 1 and case 2. For case 1 our region is bounded away from the zeros of $\cosh(\lambda z)$. The RHS of equation (3.54) is defined for all $z \in \mathbb{C}$. Hence it is relatively easy to work with this expression. For case 2, when the imaginary part of z can be an odd integer, we know that $\cosh(\lambda z) = 0$. At these points we know that the RHS of equation (3.54) is undefined. However, we can make estimates on $|\hat{g}(z)|$ for z in these rectangles. We demonstrate in the calculations that follow.

We make estimates on $\left|e^{\lambda z} + e^{-\lambda z}\right|$ on the imaginary axis:

$$\begin{aligned} \left|e^{\lambda z} + e^{-\lambda z}\right| &= \sqrt{\left(e^{\lambda x} + e^{-\lambda x}\right)^2 \left(\cos\left(\lambda y\right)\right)^2 + \left(e^{\lambda x} - e^{-\lambda x}\right)^2 \left(\sin\left(\lambda y\right)\right)^2} \\ &= \sqrt{\left(2\right)^2 \left(\cos\left(\lambda y\right)\right)^2 + \left(0\right)^2 \left(\sin\left(\lambda y\right)\right)^2} \\ &= \sqrt{4\left(\cos\left(\lambda y\right)\right)^2} \\ &= 2\left|\cos\left(\lambda y\right)\right| \end{aligned} \tag{3.55}$$

on the imaginary axis, where we have expressed $z = x + iy$, for $x, y \in \mathbb{R}$.

Figure 3.3 shows the graph of $\left|\cos\left(\lambda y\right)\right| = \left|\cos\left(\pi y/2\right)\right|$ (red curve), along with the shaded regions that we demonstrated in figure 3.2.

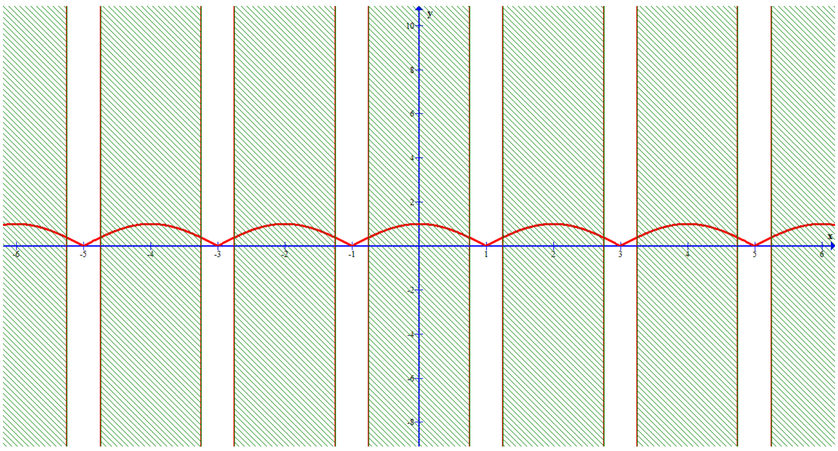

Figure 3.3: Modulus of cos and our estimates

Clearly, from figure 3.3, we can see that for case 1, that is, inside the shaded rectangles, we have that:

$$\begin{aligned} \left|\cos\left(\lambda y\right)\right| &\geq \left|\cos\left(\lambda\left(2k \pm 3/4\right)\right)\right| \\ &= \left|\cos\left(3\pi/8\right)\right|. \end{aligned} \tag{3.56}$$

It follows that:

$$|\hat{g}(z)| \leq \frac{2Ce^{A|z|}}{|e^{\lambda z} + e^{-\lambda z}|} \leq \frac{Ce^{A|z|}}{|\cos(3\pi/8)|} = C_1 e^{A|z|}. \tag{3.57}$$

Hence, we have a bound of exponential type for $\hat{g}$, for z in the region defined by $2k - 3/4 \leq |\Im(z)| \leq 2k + 3/4$, where $k \in \mathbb{Z}$.

Next we consider $|\hat{g}(i\Im(z))|$ inside the rectangles that contain the zeros of z, i.e. for $2k + 3/4 \leq |\Im(z)| \leq 2k + 5/4$, $k \in \mathbb{Z}$:

$$\begin{aligned}
|\hat{g}(i\Im(z))| &= \left|\int_{-\infty}^{\infty} g(t)\, e^{-i(i\Im(z))t}\, \mathrm{d}t\right| \\
&\leq \left|\int_{0}^{\infty} g(t)\, e^{\Im(z)t}\, \mathrm{d}t\right| + \left|\int_{-\infty}^{0} g(t)\, e^{\Im(z)t}\, \mathrm{d}t\right| \\
&\leq \left|\int_{0}^{\infty} g(t)\, e^{(\Im(z)+1)t}\, \mathrm{d}t\right| + \left|\int_{-\infty}^{0} g(t)\, e^{(\Im(z)-1)t}\, \mathrm{d}t\right| \\
&\leq |\hat{g}(i(\Im(z)+1))| + |\hat{g}(i(\Im(z)-1))| \\
&\leq C_1 e^{A|i(\Im(z)+1)|} + C_1 e^{A|i(\Im(z)-1)|} \\
&= C_2 e^{A|(\Im(z))|} + C_3 e^{A|i(\Im(z))|} \\
&\leq C_4 e^{A|z|}.
\end{aligned} \tag{3.58}$$

We note that when we go either one above or one below the strips containing the zeros of $\cosh(\lambda z)$ we are no longer inside these strips but instead we are inside the shaded strips where we have obtained a bound of exponential type, hence the result above. We have demonstrated that we have a bound of exponential type for $\hat{g}$ on the imaginary axis.

Now, for $z \in \mathbb{C}$:

$$\begin{aligned}
|\hat{g}(z)| &= \left| \int_{-\infty}^{\infty} g(t)\, e^{-izt}\, \mathrm{d}t \right| \\
&\leq \int_{-\infty}^{\infty} \left| g(t)\, e^{-ixt} e^{yt} \right| \mathrm{d}t \\
&= \int_{-\infty}^{\infty} |g(t)|\, e^{yt}\, \mathrm{d}t \\
&= \left| \int_{-\infty}^{\infty} g(t)\, e^{yt}\, \mathrm{d}t \right| = \left| \int_{-\infty}^{\infty} g(t)\, e^{-i(iy)t}\, \mathrm{d}t \right| \\
&= |\hat{g}(iy)| = |\hat{g}(i\Im(z))| \,. \qquad (3.59)
\end{aligned}$$

We have demonstrated that for either non-negative or non-positive g, we have that $|\hat{g}(z)| \leq |\hat{g}(i\Im(z))|$.

Combining the above, we have demonstrated that for $g \in F_{[\lambda,p]}$ either non-negative or non-positive g, we can find a bound of exponential type for $|\hat{g}(z)|$ for all $z \in \mathbb{C}$.

The proof above holds for g one signed. With regards to the physical problem that we are solving, it makes sense to consider g non-negative; g is a modified version of the loss modulus, which is a positive function of frequency (it is directly proportional to the dissipation that occurs in a full cycle of the deformation and this must be positive), for details see Binding [13]. However, the mathematical problem is interesting in its own right and we would like to demonstrate the above result for all $g \in F_{[\lambda,p]}$. We prove this in section 3.5.

3.5 Using Properties of Meromorphic Functions

Our previous proofs that $\hat{g}\xi_\lambda$ is of exponential type implies that $\hat{g}$ is of exponential type hold for one signed g only ($g \geq 0$ or $g \leq 0$). We would like to extend the above results to satisfy all $g \in F_{[\lambda,2]}$.

We make use of results of Rubel and Taylor [70] and Kujala [49] to demonstrate that if $|\hat{g}(z) \cdot \cosh(\lambda z)|$ is of exponential type then $|\hat{g}(z)|$ is of exponential type for all $g \in F_{[\lambda,2]}$.

Lemma 3.6. *For $g \in F_{[\lambda,2]}$, if $|\hat{g}(z) \cdot \cosh(\lambda z)|$ is of exponential type then $|\hat{g}(z)|$ is also of exponential type.*

Before we prove Lemma 3.6 we make some remarks with regards to results that we will be using in the proof.

We assume that for $\hat{g}$ an analytic function, we have the following:

$$|\hat{g}(z)| \leq \frac{Ce^{A|z|}}{|\cosh(\lambda z)|} \tag{3.60}$$

for constants $C \geq 0$ and $A > 0$ and $\lambda = \pi/2$. In the calculations above, we used this to demonstrate that $|\hat{g}(z)|$ had a bound of exponential type for almost every $z \in \mathbb{C}$. We encountered problems when $\cosh(\lambda z) = 0$ since the RHS of equation (3.60) is undefined when $\cosh(\lambda z) = 0$, that is when $z = \pm i, \pm 3i, \pm 5i, \pm 7i, \cdots$.

We define the function $f(z)$ as:

$$f(z) = \frac{Ce^{A|z|}}{|\cosh(\lambda z)|}. \tag{3.61}$$

We note that $f(z)$ is a meromorphic function, where we define a meromorphic function as:

Definition 3.7. *A function f on an open set Ω is meromorphic if there exists a discrete set of isolated points $S = \{z : z \in \Omega\}$ such that f is holomorphic on $\Omega \setminus S$ and has poles at each $z \in S$.*

Next we refer to some important results of Rubel and Taylor [70]. The main result that we will make use of in this discussion is the following:

Definition 3.8. *A meromorphic function f is said to be of finite ρ-type if there exists a constant b such that $T(r, f) \leq b\rho(r)$, where $T(r, f)$ denotes the Nevanlinna Characteristic, and ρ a growth function.*

We note that Rubel and Taylor actually use λ-type in their work, but since we have already used $\lambda = \pi/2$ we shall use ρ-type when referring to the work of Rubel and Taylor [70].

Rubel and Taylor [70] define the growth function $\rho(r)$ as follows:

Definition 3.9. *A growth function $\rho(r)$ is a function defined for $0 \leq r < \infty$ that is positive, nondecreasing and continuous.*

Furthermore, we note that the Nevanlinna Characteristic, $T(r, f)$, is defined (see Ablowitz and Halburd [1] for more details) as follows:

$$T(r, f) = m(r, f) + N(r, f). \tag{3.62}$$

The function $m(r, f)$, known as the proximity function, is defined as:

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| \mathrm{d}\theta, \tag{3.63}$$

where $\log^+(x) = \max\{0, \log x\}$. The integrated counting function, $N(r, f)$, is defined to be:

$$N(r, f) = N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} \mathrm{d}t + n(0, f) \log r, \tag{3.64}$$

where $n(r, f)$ is defined to be the number of poles of f in the disk $|z| < r$.

Rubel and Taylor [70] continue by making the following remark with regards to an entire function of finite ρ-type:

Remark 3.10. *An entire function f is of finite ρ-type if and only if there are positive constants α and R such that $|f(z)| \leq \exp\{\alpha\rho(|z|)\}$ for all $z \in \mathbb{C}$ with $|z| > R$.*

In order for us to demonstrate that $f(z)$ is of finite ρ-type, we need to satisfy Definition 3.8, that is, $T(r, f) \leq b\rho(r)$.

We now have all the necessary results needed to prove Lemma 3.6.

The proof goes as follows: we demonstrate that $T(r, \hat{g}) = m(r, \hat{g}) \leq m(r, f) \leq \alpha\rho(r)$.

It follows that $\hat{g}$ is of finite ρ-type and since we are assuming that $\hat{g}$ is an entire function, it follows from the remark of Rubel and Taylor [70] that $|\hat{g}(z)| \leq \exp\{\alpha\rho(|z|)\}$. Given the form of ρ, $\hat{g}$ is of exponential type.

Proof of Lemma 3.6: We consider $T(r, \hat{g}) = m(r, \hat{g}) + N(r, \hat{g})$. Since $\hat{g}$ does not have any poles we can deduce immediately that $N(r, \hat{g}) = 0$, hence, we need only consider $m(r, \hat{g})$:

$$m(r, \hat{g}) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|\hat{g}\left(re^{i\theta}\right)\right| \mathrm{d}\theta. \tag{3.65}$$

Since $\log^+(x) = \max\{0, \log x\}$, is an increasing function if follows from equation (3.60) that we can write equation (3.65) as:

$$m(r, \hat{g}) \leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|\frac{Ce^{A\left|re^{i\theta}\right|}}{\cosh\left(\lambda re^{i\theta}\right)}\right| \mathrm{d}\theta. \tag{3.66}$$

Next, we split equation (3.66) into the two integrals:

$$\begin{aligned} m(r, \hat{g}) &\leq \frac{1}{2\pi} \int_0^{2\pi} \Big|_{\{\theta:|f(r,\theta)|\leq 1\}} \log^+ |f(r, \theta)| \, \mathrm{d}\theta + \frac{1}{2\pi} \int_0^{2\pi} \Big|_{\{\theta:|f(r,\theta)|>1\}} \log^+ |f(r, \theta)| \, \mathrm{d}\theta \\ &= 0 + \frac{1}{2\pi} \int_0^{2\pi} \Big|_{\{\theta:|f(r,\theta)|>1\}} \log |f(r, \theta)| \, \mathrm{d}\theta. \end{aligned} \tag{3.67}$$

Now, in the work of Rubel and Taylor [70] they do not specify to what base their log function is, so assuming it is to the base d, we note the following relationship that we can make use of:

$$\log_d L = \frac{\log_j L}{\log_j d}.$$

We will be taking $\log_j$ to be the natural logarithm ln. Hence, we can express equation (3.67) as:

$$\begin{aligned} m(r,\hat{g}) &\leq \frac{1}{2\pi}\int_0^{2\pi}\Big|_{\{\theta:|f(r,\theta)|>1\}} \frac{1}{\ln(d)} \ln|f(r,\theta)|\,\mathrm{d}\theta \\ &\leq \frac{1}{2\pi\ln(d)}\int_0^{2\pi}\Big|_{\{\theta:|f(r,\theta)|>1\}} \ln\left|Ce^{A|re^{i\theta}|}\right| - \ln\left|\cosh\left(\lambda re^{i\theta}\right)\right|\,\mathrm{d}\theta. \quad (3.68)\end{aligned}$$

For the second part of the integrand of (3.68), discarding those θ for which $\ln\left|\cosh\left(\lambda re^{i\theta}\right)\right|$ is positive, we have:

$$\begin{aligned} m(r,\hat{g}) &\leq \frac{1}{2\pi\ln(d)}\int_0^{2\pi} \ln\left|Ce^{A|re^{i\theta}|}\right|\,\mathrm{d}\theta \\ &- \frac{1}{2\pi\ln(d)}\int_0^{2\pi}\Big|_{\{\theta:|\cosh(\lambda re^{i\theta})|<1\}} \ln\left|\cosh\left(\lambda re^{i\theta}\right)\right|\,\mathrm{d}\theta \\ &= \frac{1}{2\pi\ln(d)}\int_0^{2\pi} \ln(C) + Ar\,\mathrm{d}\theta \\ &- \frac{1}{2\pi\ln(d)}\int_0^{2\pi}\Big|_{\{\theta:|\cosh(\lambda re^{i\theta})|<1\}} \ln\left|\cosh\left(\lambda re^{i\theta}\right)\right|\,\mathrm{d}\theta. \quad (3.69)\end{aligned}$$

Next, we make use of the following lemma:

Lemma 3.11. *For values of θ such that $\left|\cosh\left(\lambda re^{i\theta}\right)\right| < 1$, the following integral is finite:*

$$
\begin{aligned}
& - \int_0^{2\pi} \Big|_{\left\{\theta : \left|\cosh\left(\lambda r e^{i\theta}\right)\right| < 1\right\}} \ln\left|\cosh\left(\lambda r e^{i\theta}\right)\right| \, \mathrm{d}\theta \\
\leq & - \int_0^{2\pi} \Big|_{\left\{\theta : \left|\cosh\left(\lambda e^{i\theta}\right)\right| < 1\right\}} \ln\left|\cosh\left(\lambda e^{i\theta}\right)\right| \, \mathrm{d}\theta \\
\equiv & \ \tilde{W} < \infty. \qquad (3.70)
\end{aligned}
$$

The proof of Lemma 3.11 is in section 7.5 in the appendix. We deduce that:

$$
\begin{aligned}
m(r, \hat{g}) & \leq \frac{\ln(2C) + Ar}{\ln(d)} + \frac{\tilde{W}}{2\pi \ln(d)} \\
& = \frac{1}{\ln(d)} \left[\ln(C) + Ar + \frac{\tilde{W}}{2\pi} \right]. \qquad (3.71)
\end{aligned}
$$

That is, we have demonstrated that $T(r, \hat{g}) = m(r, \hat{g}) \leq \frac{\rho(r)}{\ln(d)}$, where $\rho(r) = \ln(2C) + Ar + \frac{\tilde{W}}{2\pi}$, which is positive, non-decreasing and continuous, as required.

It follows from definition 3.8 and remark 3.10 that:

$$
\begin{aligned}
|\hat{g}(z)| & \leq \exp\left\{ \frac{\rho(A|z|)}{\ln(d)} \right\} = \exp\left\{ \frac{\tilde{C} + A|z|}{\ln(d)} \right\} \\
& = \tilde{C}_d \exp\left\{ \frac{A|z|}{\ln(d)} \right\}, \qquad (3.72)
\end{aligned}
$$

where $\tilde{C}_d = \exp\left\{\tilde{C}/\ln(d)\right\}$ and $\tilde{C} = \ln(2C) + \frac{\tilde{W}}{2\pi}$. That is, $\hat{g}$ has a bound of exponential type.

Combining the above, we have demonstrated that for all $g \in F_{[\lambda,2]}$, if $|\xi_\lambda \cdot \hat{g}|$ has a bound of exponential type then $|\hat{g}|$ has a bound of exponential type.

In the next section we will use the results of this chapter to prove Theorem 3.1.

3.6 Properties of the Relaxation Spectrum

We remind the reader of Theorem 3.1 that we introduced at the beginning of the chapter:

Theorem 3.1. *For a function g belonging to the space $F_{[\lambda,2]}$, if $\hat{g}$ is entire such that $|\hat{g}(z) \cdot \cosh(\lambda z)| = \left|\hat{h}\right|$ is of exponential type, then $g = 0$ and hence $h = 0$.*

Proof:

Now, if we refer to the Paley-Wiener Theorem, Theorem 3.2 at the beginning of the chapter, we see that the Fourier transform of an entire function f of exponential type can be written in the form:

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\gamma}^{\gamma} F(t) \mathrm{e}^{izt} \, \mathrm{d}t \tag{3.73}$$

for all z.

Since we have proved that $\hat{g}\xi_\lambda$ is of exponential type implies that $\hat{g}$ is of exponential type, it follows from the Paley-Wiener Theorem that $\mathscr{F}^{-1}[\hat{g}] = g$ has compact support. Using results of section 3.3 we have seen that the only function in $F_{[\lambda,2]}$ with compact support is the zero function. We deduce that $g = 0$ and hence $h = 0$.

This completes the proof.

3.7 Discussion

At the beginning of this chapter we noted that there is an alternative method, using Paley-Wiener type theorems, for obtaining an interval of compact support for a function, given certain properties of its Fourier transform. The original work of Loy, Newbury, Andersen and Davies [50] made use of this method in addition to their method that we demonstrated in Chapter 2. They obtained an interval of support for the relaxation spectrum using this method. They made use of the assumption that g was compactly supported. We have demonstrated in this chapter that Renardy's claim is correct; that for $g \in F_{[\lambda,p]}$ with g compactly supported, g must be the zero function.
In the revised work by Loy, Davies and Anderssen [51] they make modifications to their methods, to account for g not having compact support and still obtain the same interval of support for h as in their previous work. They do not however refer to the Paley-Wiener Theorem in their revised work.

We have demonstrated in this chapter, by making different modifications to $\hat{g}$, the difficulty in trying to obtain a bound of exponential type for $\hat{g}\cosh$, which is required to apply the Paley-Wiener Theorem. We have proven that if $\left|\hat{h}\right| = \left|\mathscr{F}^{-1}\left[\hat{g}\xi_\lambda\right]\right|$ for $g \in F_{[\lambda,2]}$, has a bound of exponential type then $g = 0$ and subsequently $h = 0$.

In the next chapter we demonstrate stronger results to those in sections 3.3, 3.4 and 3.5 where we work with functions in the space of tempered distributions.

Chapter 4

Non-Compactness of Support Using Paley-Wiener in $\mathscr{S}'(\mathbb{R})$

In the previous chapter we demonstrated the difficulty in obtaining results of compact support for the relaxation spectrum. We proved that it is impossible to use the Paley-Wiener Theorem to deduce that $h = \mathscr{F}^{-1}[\hat{g}\xi_\lambda]$ has compact support since $|\hat{g}\xi_\lambda|$ can only have a bound of exponential type if $g = 0$, which would imply that $\hat{g}$ and hence h are also zero.

We are able to prove a stronger result in Schwartz space (space of tempered distributions), which allows us to deduce that h cannot have compact support for $g \in F_{[\lambda,\mathscr{S}']}$,(which we will define in a moment), where $\mathscr{S}'$ is the space of tempered distributions. From this, we also deduce that h cannot have compact support for $g \in F_{[\lambda,p]}$.

In the following section we introduce some important definitions and theorems relating to the space of tempered distributions.

4.1 Properties and Results Relating to the Space of Tempered Distributions $\mathscr{S}'$

We introduce the space of rapidly decaying test functions $\mathscr{S}(\mathbb{R})$ and the space of tempered distributions $\mathscr{S}'(\mathbb{R})$, which we will be making use of in this chapter:

Definition 4.1. *The Schwartz Class*

A C^∞ function is of class $\mathscr{S}(\mathbb{R})$ if φ and all its partial derivatives are rapidly decreasing.

Definition 4.2. *Tempered (Schwartz) Distributions*

The space of all tempered distributions on $\mathbb{R}$ denoted by $\mathscr{S}'(\mathbb{R})$ is the dual space of $\mathscr{S}(\mathbb{R})$ consisting of continuous linear forms on $\mathscr{S}(\mathbb{R})$. A linear form f is continuous if and only if there is a constant $C \geq 0$ and a nonnegative integer N such that

$$|\langle f, \varphi \rangle| \leq C \sum_{|\alpha|,|\beta| \leq N} \sup \left| x^\alpha D^\beta \varphi \right|, \tag{4.1}$$

for $\varphi \in \mathscr{S}(\mathbb{R})$.

For more detail see Friedlander and Joshi [29] and Strichartz [76].

We define the space $F_{[\lambda, \mathscr{S}']}$ as:

$$F_{[\lambda, \mathscr{S}']} = \left\{ g \in L^1(\mathbb{R}) \mid \xi_\lambda \cdot \hat{g} \in \mathscr{S}'(\mathbb{R}) \right\}. \tag{4.2}$$

We introduce an important theorem (taken from Friedlander and Joshi [29]) that we will be making use of in this chapter:

Theorem 4.3. *Paley-Wiener-Schwartz Theorem*

Let A be a positive real number. A function $F(z)$ which is analytic on $\mathbb{C}$ is the Fourier-Laplace transform of a distribution $f \in \mathscr{D}'(\mathbb{R})$ supported in $|x| \leq A$ if and only if there is an estimate:

$$|F(z)| \leq C(1+|z|)^N e^{A|\Im(z)|} \quad \forall z \in \mathbb{C}^n,$$

for some constants $C, N \geq 0$.

For more detail see Friedlander and Joshi [29].

We note a few important points with regards to Theorem 4.3; firstly, that the space $L^2(\mathbb{R})$ is a subspace of $\mathscr{S}'(\mathbb{R})$ and secondly, that the Paley-Wiener-Schwartz Theorem is an if and only if theorem, that is, if a bound of exponential type does not exist, then the corresponding Fourier transform does not have compact support. The Paley-Wiener Theorem says that if a function, satisfying all of the criteria of the theorem has compact support *then* its corresponding Fourier transform will have a bound of exponential type.

We state the main results of this chapter in the form of a theorem and corollary:

Theorem 4.4. *For a function h belonging to the space $\mathscr{S}'(\mathbb{R})$, where $h = \mathscr{F}^{-1}[\hat{g}\xi_\lambda]$ for $g \in F_{[\lambda,\mathscr{S}']}$, if h has compact support then $h \equiv 0$.*

Corollary 4.5. *If $h \in L^q(\mathbb{R})$ has compact support, where $h = \mathscr{F}^{-1}[\hat{g}(r) \cdot \cosh(\lambda r)]$ for*

$g \in F_{[\lambda,p]}$ *and* $\hat{g}(r) \cdot \cosh(\lambda r) \in L^p(\mathbb{R})$*, then* $h = 0$*.*

Since $L^q(\mathbb{R})$ is a subspace of $\mathscr{S}'(\mathbb{R})$, we can deduce that if h belongs to $L^q(\mathbb{R})$ (it also belongs to $\mathscr{S}'(\mathbb{R})$) and has compact support then it must be the zero function, using the Paley-Wiener-Schwartz Theorem and Theorem 4.4.

In proving the main results of this chapter, we will be making use of similar results in Chapter 3. An important result that we will be making use of is stated in the following proposition:

Proposition 4.6. *For a function g belonging to the space $F_{[\lambda,\mathscr{S}']}$, g is an analytic function. Furthermore, if g has compact support then g must be the zero function.*

We refer the reader to equations (3.34) to (3.50) in section 3.3 of Chapter 3 and note that we can apply the same set of arguments to g belonging to $F_{[\lambda,\mathscr{S}']}$ except at equation (3.36):

$$\begin{aligned} |g(z)| &= |g(t+is)| \\ &\leq \int_{-\infty}^{\infty} |\hat{g}(r)\exp(ir(t+is))|\,\mathrm{d}r \\ &\leq 2\int_{-\infty}^{\infty} e^{|r|(|s|-\lambda)}\,|\xi_\lambda(r)\,\hat{g}(r)|\,\mathrm{d}r, \end{aligned} \tag{4.3}$$

where we note that equation (4.3) is finite for $g \in F_{[\lambda,\mathscr{S}']}$ since $|s| - \lambda$ is always negative it follows that $e^{|r|(|s|-\lambda)}$ is a function of rapid decay, i.e. $e^{|r|(|s|-\lambda)}$ belongs to $\mathscr{S}(\mathbb{R})$. Furthermore, the function $|\xi_\lambda(r)\,\hat{g}(r)|$ is defined as belonging to $\mathscr{S}'(\mathbb{R})$. Hence, we deduce from the properties of distributions that the integral in equation (4.3) is finite.

It follows that g is bounded.

The following calculations in that section hold and hence, we have that proposition 4.6 is true.

4.2 Results of Non-compactness of Support using the Paley-Wiener-Schwartz Theorem

In order to prove Theorem 4.4 we must first prove the following lemma:

Lemma 4.7. *For* $g \in F_{[\lambda,\mathscr{S}']}$ *and either* $g \geq 0$ *or* $g \leq 0$ *(g one signed), if* $|\hat{g}(z) \cdot \cosh(\lambda z)|$ *is of exponential type then* $|\hat{g}(z)|$ *is also of exponential type.*

Before we begin the proof, we make some remarks with regards to the Fourier transform of a Schwartz (tempered) distribution.

The Fourier transform of a tempered distribution, w is defined as:

$$\langle \hat{w}, \varphi \rangle = \langle w, \hat{\varphi} \rangle \tag{4.4}$$

for $w \in \mathscr{S}'(\mathbb{R})$ and $\varphi \in \mathscr{S}(\mathbb{R})$. We note also that $\hat{w} \in \mathscr{S}'(\mathbb{R})$ and $\hat{\varphi} \in \mathscr{S}(\mathbb{R})$, that is the Fourier transform preserves the class $\mathscr{S}(\mathbb{R})$ and the Fourier transform preserves the space of tempered distributions, for more details see Chapter 3 of Strichartz [76].

If we consider our function $\hat{g}$:

$$\begin{aligned} \hat{g}(z) &= \int_{-\infty}^{\infty} g(t)\, e^{-izt}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} g(t)\, e^{-ixt} e^{yt}\,\mathrm{d}t. \end{aligned} \tag{4.5}$$

Now, if $g(t)\, e^{yt} \in L^1(\mathbb{R})$ then equation (4.5) is finite, and hence we can make estimates on $\hat{g}$ using the equation above. However, if $g(t)\, e^{yt}$ is not an integrable function, then we must use the definition in equation (4.4) in order to make estimates on $\hat{g}$. We must have that $\hat{g}$ exists for us to be able to apply the Paley-Wiener-Schwartz Theorem. Furthermore, we saw in the previous chapter, for calculations of g in the space $F_{[\lambda,2]}$, that we had to assume that $g(t)\, e^{yt} \in L^1(\mathbb{R})$ in order to have the existence of $\hat{g}$. Hence, it makes sense to assume that $g(t)\, e^{yt}$ is integrable. It follows from the equation (4.4) that $g(t)\, e^{yt}$ also belongs to the space of tempered distributions. For more details on tempered distributions see Chapter 8 of Friedlander and Joshi [29].

Now we begin the Proof of Lemma 4.7:

We begin by assuming that we can find a bound of exponential type for $|\xi_\lambda \cdot \hat{g}| = |\cosh(\lambda z) \cdot \hat{g}(z)|$. That is, assume we have constants $C \geq 0$ and $A > 0$, such that:

$$|\cosh(\lambda z) \cdot \hat{g}(z)| \leq C(1 + |z|)^N e^{A|\Im(z)|}. \tag{4.6}$$

We note here that there is a difference in the bound of exponential type for $|\cosh(\lambda z) \cdot \hat{g}(z)|$

when $g \in F_{[\lambda,\mathscr{S}']}$ compared to the one we introduced in equation (3.53) when $g \in F_{[\lambda,2]}$. We refer to Figure (3.2) and note that we will be considering the same two cases as those in Chapter 3, i.e. the rectangles containing the even integers on the imaginary line and the rectangles containing the odd integers on the imaginary line.

If we consider case 1, where we work with the shaded rectangles in Figure (3.2), we can use the results in Chapter 3, see equation (3.57), to note that:

$$|\hat{g}(z)| \leq \frac{2C(1+|z|)^N e^{A|\Im(z)|}}{|e^{\lambda z}+e^{-\lambda z}|} \leq \frac{C(1+|z|)^N e^{A|\Im(z)|}}{|\cos(3\pi/8)|} = C_1 (1+|z|)^N e^{A|\Im(z)|} \tag{4.7}$$

on the imaginary axis, away from the zeros. Now, we would like to show that we can find a bound of exponential type for $\hat{g}(z)$, when z is in the rectangles containing the imaginary odd integers. We refer the reader to equation (3.58) and note that we can use very similar arguments to demonstrate that for either non negative or non positive g, we have that:

$$\begin{aligned}
|\hat{g}(i\Im(z))| &\leq \left|\int_{-\infty}^{\infty} g(t)\, e^{-i(i\Im(z))t}\,\mathrm{d}t\right| \\
&\leq \left|\int_{0}^{\infty} g(t)\, e^{\Im(z)t}\,\mathrm{d}t\right| + \left|\int_{-\infty}^{0} g(t)\, e^{\Im(z)t}\,\mathrm{d}t\right| \\
&\leq \left|\int_{0}^{\infty} g(t)\, e^{(\Im(z)+1)t}\,\mathrm{d}t\right| + \left|\int_{-\infty}^{0} g(t)\, e^{(\Im(z)-1)t}\,\mathrm{d}t\right| \\
&\leq |\hat{g}(i(\Im(z)+1))| + |\hat{g}(i(\Im(z)-1))| \\
&= C_1(1+|\Im(z)+1|)^N e^{A|\Im(i\Im(z)+i)|} + C_1(1+|\Im(z)-1|)^N e^{A|\Im(i\Im(z)-i)|} \\
&\leq C_2(2+|\Im(z)|)^N e^{A(|\Im(z)|+1)} \\
&\leq C_3(1+|z|)^N e^{A|\Im(z)|}.
\end{aligned} \tag{4.8}$$

Furthermore, using the same arguments as those in equation (3.59), we demonstrate that $|\hat{g}(z)| \leq |\hat{g}(i\Im(z))|$ for either non-negative or non-positive $g \in F_{[\lambda,\mathscr{S}']}$.

Combining the above, we have demonstrated that for g non-negative, or non-positive (g one signed) and for $g \in F_{[\lambda,\mathscr{S}']}$ that if $|\hat{g}(z) \cdot \cosh(\lambda z)|$ has a bound of exponential type, then $|\hat{g}(z)|$ also has a bound of exponential type. This completes the proof.

The proof above holds for g one signed. With regards to the physical problem that we are solving, it makes sense to consider g nonnegative; g is a modified version of the loss modulus, which is a positive function of frequency (it is directly proportional to the dissipation that occurs in a full cycle of the deformation and this must be positive). However, the mathematical problem is interesting in its own right and we would like to demonstrate the above result for all $g \in F_{[\lambda,\mathscr{S}']}$. We will prove this result in section 4.3.

4.3 Using Properties of Meromorphic Functions

In this section we will perform similar calculations to those in section 3.5 in Chapter 3 to prove the following lemma:

Lemma 4.8. *For $g \in F_{[\lambda,\mathscr{S}']}$, if $|\hat{g}(z) \cdot \cosh(\lambda z)|$ is of exponential type then $|\hat{g}(z)|$ is also of exponential type.*

This is Lemma 4.7 without the constraint of g being one signed.

We demonstrate that if $|\hat{g}(z) \cosh(\lambda z)|$ is of exponential type, then it follows that $|\hat{g}(z)|$ is also of exponential type. We assume that we have the following:

$$|\hat{g}(z)| \leq \frac{C(1+|z|)^N e^{A|\Im(z)|}}{|\cosh(\lambda z)|} \tag{4.9}$$

for constants $C \geq 0$ and $A > 0$ and $\lambda = \pi/2$. We used this to demonstrate that $|\hat{g}(z)|$ had a bound of exponential type for all $z \in \mathbb{C}$. We encountered problems when $\cosh(\lambda z) = 0$ since the RHS of equation (4.9) is undefined when $\cosh(\lambda z) = 0$, that is when $z = \pm i, \pm 3i, \pm 5i, \pm 7i, \cdots$.

We define the function $\mu(z)$ as:

$$\mu(z) = \frac{C(1+|z|)^N e^{A|\Im(z)|}}{|\cosh(\lambda z)|} \tag{4.10}$$

and we note that $\mu(z)$ is a meromorphic function.

We remind the reader that we will be making use of the results of Rubel and Taylor [70] that we introduced in section of Chapter 3.

We follow almost the same set of calculations as those in section of Chapter 3, but with slight differences. We demonstrate that $T(r, \hat{g}) = m(r, \hat{g}) \leq m(r, \mu) \leq \alpha\rho(r)$. It follows that $\hat{g}$ is of finite ρ-type. Since we are assuming that $\hat{g}$ is an entire function, it follows from the remark of Rubel and Taylor [70] that $|\hat{g}(z)| \leq \exp\{\alpha\rho(|z|)\}$. However, we have already seen that this bound of exponential is not in the correct form to be able to apply the Paley-Wiener-Schwartz Theorem. For the Paley-Wiener-Schwartz Theorem, we require a bound of exponential type of the form $(1+|z|)^M \exp\{\gamma|\Im(z)|\}$, for some constant γ. We have to make additional calculations to get our bound of exponential type in the correct form by splitting the complex plane into regions where $|\Im(z)| \geq |\Re(z)|$ and $|\Im(z)| < |\Re(z)|$. We demonstrate below.

Proof of Lemma 4.8 We consider $T(r, \hat{g}) = m(r, \hat{g})$, since $\hat{g}$ does not have any poles (it is entire) we can deduce immediately that $N(r, \hat{g}) = 0$, hence, we need only consider $m(r, \hat{g})$:

$$m(r, \hat{g}) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \hat{g}\left(re^{i\theta}\right) \right| \, \mathrm{d}\theta. \tag{4.11}$$

Since $\log^+(x) = \max\{0, \log x\}$, is an increasing function if follows from equation (4.9) that we can write equation (4.10) as:

$$m(r, \hat{g}) \leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \mu\left(re^{i\theta}\right) \right| \, \mathrm{d}\theta. \tag{4.12}$$

Next, we split equation (4.12) into the two integrals:

$$\begin{aligned} m(r, \hat{g}) &\leq \frac{1}{2\pi} \int_0^{2\pi} \Big|_{\theta:\mu(r,\theta)\leq 1} \log^+ \left| \mu\left(re^{i\theta}\right) \right| \, \mathrm{d}\theta + \frac{1}{2\pi} \int_0^{2\pi} \Big|_{\theta:\mu(r,\theta)>1} \log^+ \left| \mu\left(re^{i\theta}\right) \right| \, \mathrm{d}\theta \\ &= 0 + \frac{1}{2\pi} \int_0^{2\pi} \Big|_{\theta:\mu(r,\theta)>1} \log \left| \mu\left(re^{i\theta}\right) \right| \, \mathrm{d}\theta. \end{aligned} \tag{4.13}$$

Changing the log function into the natural logarithm as we did above we can express equation (4.13) as:

$$\begin{aligned}
m(r,\hat{g}) &\leq \frac{1}{2\pi}\int_0^{2\pi}\Big|_{\{\theta:|\mu(r,\theta)|>1\}} \frac{1}{\ln(d)} \ln|\mu(r,\theta)|\,\mathrm{d}\theta \\
&\leq \frac{1}{2\pi\ln(d)}\int_0^{2\pi}\Big|_{\{\theta:|\mu(r,\theta)|>1\}} \ln\left|C\left(1+\left|re^{i\theta}\right|\right)^N e^{A\left|\Im\left(re^{i\theta}\right)\right|}\right| - \ln\left|\cosh\left(\lambda re^{i\theta}\right)\right|\,\mathrm{d}\theta \\
&\leq \frac{1}{2\pi\ln(d)}\int_0^{2\pi} \ln\left|C\left(1+\left|re^{i\theta}\right|\right)^N e^{A\left|\Im\left(re^{i\theta}\right)\right|}\right|\,\mathrm{d}\theta \\
&- \frac{1}{2\pi\ln(d)}\int_0^{2\pi}\Big|_{\theta:\left|\cosh\left(\lambda re^{i\theta}\right)\right|<1} \ln\left|\cosh\left(\lambda re^{i\theta}\right)\right|\,\mathrm{d}\theta. \qquad (4.14)
\end{aligned}$$

For the second part of the integrand of equation (4.14), discarding those θ for which $\ln\left|\cosh\left(\lambda re^{i\theta}\right)\right|$ is positive, we have:

$$\begin{aligned}
m(r,\hat{g}) &\leq \frac{1}{2\pi\ln(d)}\int_0^{2\pi} \ln\left|C\left(1+\left|re^{i\theta}\right|\right)^N e^{A\left|\Im\left(re^{i\theta}\right)\right|}\right|\,\mathrm{d}\theta \\
&- \frac{1}{2\pi\ln(d)}\int_0^{2\pi}\Big|_{\theta:\left|\cosh\left(\lambda re^{i\theta}\right)\right|<1} \ln\left|\cosh\left(\lambda re^{i\theta}\right)\right|\,\mathrm{d}\theta. \qquad (4.15)
\end{aligned}$$

We note here that we can use the same set of arguments and calculations as those in Chapter 3 (equations (7.55) to (7.59)), to express equation (4.15) as:

$$\begin{aligned}
m(r,\hat{g}) &\leq \frac{1}{2\pi\ln(d)}\int_0^{2\pi} \ln\left|C\left(1+\left|re^{i\theta}\right|\right)^N e^{A\left|\Im\left(re^{i\theta}\right)\right|}\right|\,\mathrm{d}\theta \\
&- \frac{\ln(1/2)}{\ln(d)} + \frac{\tilde{W}}{2\pi\ln(d)}, \qquad (4.16)
\end{aligned}$$

where $\tilde{W}$ is a positive constant. Integrating with respect to θ, we obtain:

$$\begin{aligned}
m(r,\hat{g}) &\leq \frac{1}{\ln(d)}\left[\ln(2C)+\ln(1+r)^N+\frac{\tilde{W}}{2\pi}+\frac{2Ar}{\pi}\right] \\
&\leq \frac{1}{\ln(d)}\left[\ln(2C)+\ln(1+r)^N+\frac{\tilde{W}}{2\pi}+2Ar\right] \\
&= \frac{1}{\ln(d)}\left[\sigma+\ln(1+r)^N+2Ar\right] \qquad (4.17)
\end{aligned}$$

for all $z \in \mathbb{C}$. Note that $\sigma = \ln(2C) + \tilde{W}/2\pi$.

We have demonstrated that:

$$T(r,\hat{g}) \leq \frac{1}{\ln(d)}\rho(r), \qquad (4.18)$$

where

$$\rho(r) = \sigma + \ln(1+r)^N + 2Ar, \qquad (4.19)$$

where $\rho(r)$ is positive, nondecreasing and continuous as required.

It follows that $\hat{g}$ is of finite ρ-type and hence it follows from Rubel and Taylor [70] (Remark 3.10), that we can write:

$$\begin{aligned}
|\hat{g}(z)| &\leq \exp\left\{\frac{1}{\ln(d)}\rho(|z|)\right\} \\
&= \exp\left\{\frac{1}{\ln(d)}\left(\sigma + \ln(1+|z|)^N + 2A|z|\right)\right\} \\
&= K\exp\left\{\frac{1}{\ln(d)}\left(\ln(1+|z|)^N\right)\right\}\exp\{M|z|\} \\
&\leq K\exp\left\{\ln(1+|z|)^N\right\}\exp\{M|z|\} \\
&= K(1+|z|)^N\exp\{M|z|\}, \qquad (4.20)
\end{aligned}$$

where $K = \exp\{\sigma/\ln(d)\}$, $M = 2A/\ln(d)$. We note that we have taken $1/(\ln(d)) \leq 1$ since the logarithm in Rubel and Taylors work was either to the base 10 or to the base e, hence d is either 10 or e.

As we mentioned above, this bound is not in the correct form to be able to apply the Paley-Wiener-Schwartz Theorem to $\hat{g}$. We overcome this by considering two cases.

1. Case 1. We consider the values in the complex plane where $|\Im(z)| \geq |\Re(z)|$.

2. Case 2. We consider the values in the complex plane where $|\Im(z)| < |\Re(z)|$.

See Figure 4.1 where the shaded regions represent case 1, where $|\Im(z)| \geq |\Re(z)|$.

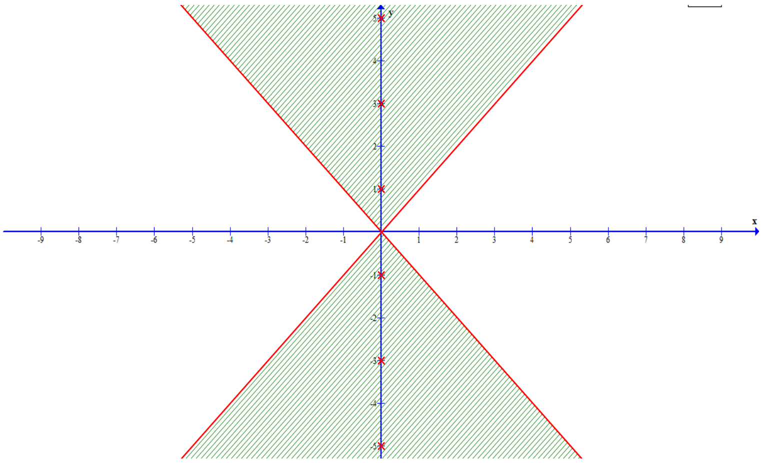

Figure 4.1: Complex Plane with region $|\Im(z)| \geq |\Re(z)|$ shaded

We consider case 1 and note that when $|\Im(z)| \geq |\Re(z)|$, we have that:

$$\begin{aligned} |z| &= \sqrt{[\Re(z)]^2 + [\Im(z)]^2} \\ &\leq \sqrt{[\Im(z)]^2 + [\Im(z)]^2} \\ &= \sqrt{2[\Im(z)]^2} \\ &= \sqrt{2}\,|\Im(z)|\,. \end{aligned}$$

Substituting this back into equation (4.20), we get that:

$$\begin{aligned} |\hat{g}(z)| &\leq \exp\{K\}(1+|z|)^N \exp\{M|z|\} \\ &\leq \exp\{K\}(1+|z|)^N \exp\left\{\sqrt{2}M|\Im(z)|\right\}, \end{aligned} \tag{4.21}$$

which is now in the correct form to be able to apply the Paley-Wiener-Schwartz Theorem.

Now, for case 2, we can see from Figure 4.1 that we are no longer in the region where $\mu(z)$ has poles (red crosses in Figure 4.1). If we refer back to equation (4.9):

$$|\hat{g}(z)| \leq \frac{C(1+|z|)^N e^{A|\Im(z)|}}{|\cosh(\lambda z)|}. \tag{4.22}$$

We make estimates on $|\cosh(\lambda z)|$, where $x = \Re(z)$ and $y = \Im(z)$:

$$\begin{aligned}
|\cosh(\lambda z)| &= \frac{1}{2}\left|e^{\lambda z} + e^{-\lambda z}\right| \\
&= \sqrt{\left(e^{\lambda x} + e^{-\lambda x}\right)^2 [\cos(\lambda y)]^2 + \left(e^{\lambda x} - e^{-\lambda x}\right)^2 [\sin(\lambda y)]^2} \\
&= \sqrt{\left(e^{2\lambda x} + e^{-2\lambda x} + 2\right)[\cos(\lambda y)]^2 + \left(e^{2\lambda x} + e^{-2\lambda x} - 2\right)^2 [\sin(\lambda y)]^2} \\
&= \sqrt{e^{2\lambda x} + e^{-2\lambda x} + 2\left\{[\cos(\lambda y)]^2 - [\sin(\lambda y)]^2\right\}} \\
&= \sqrt{e^{2\lambda x} + e^{-2\lambda x} + 2\left\{[2\cos(\lambda y)]^2 - 1\right\}} \\
&\geq \sqrt{e^{2\lambda y} + e^{-2\lambda y} + 2\left\{[2\cos(\lambda y)]^2 - 1\right\}} \\
&> 2,
\end{aligned} \tag{4.23}$$

where we have made use of the fact that $e^{2\lambda x} + e^{-2\lambda x} > e^{2\lambda y} + e^{-2\lambda y}$, since $|\Re(z)| > |\Im(z)|$.

We note that $\sqrt{e^{2\lambda y} + e^{-2\lambda y} + 2\left\{[2\cos(\lambda y)]^2 - 1\right\}}$ has a minimum, which can be calculated by differentiation. Referring back to equation (4.23):

$$|\cosh(\lambda z)| > 2.$$

It follows that:

$$\begin{aligned} |\hat{g}(z)| &\leq \frac{C(1+|z|)^N e^{A|\Im(z)|}}{|\cosh(\lambda z)|} \\ &\leq \frac{C}{2}(1+|z|)^N e^{A|\Im(z)|}. \end{aligned} \tag{4.24}$$

We have demonstrated in this section that for all g in either of the spaces $g \in F_{[\lambda,p]}$ or $g \in F_{[\lambda,\mathscr{S}']}$, such that $|\hat{g}(z) \cdot \cosh(\lambda z)|$ has a bound of exponential type, that $|\hat{g}(z)|$ has a bound of exponential type for all $z \in \mathbb{C}$.

Now we remind the reader of Theorem 4.4 we introduced at the beginning of the chapter:

Theorem 4.4. *For a function h belonging to the space $\mathscr{S}'(\mathbb{R})$, where $h = \mathscr{F}^{-1}[\hat{g}\xi_\lambda]$ for $g \in F_{[\lambda,\mathscr{S}']}$, if h has compact support then $h \equiv 0$.*

Proof:

Suppose $h = \mathscr{F}^{-1}[\hat{g} \cdot \xi_\lambda]$ has compact support, then it follows from the Paley-Wiener-Schwartz Theorem that $|\hat{g} \cdot \xi_\lambda|$ is of exponential type. We can deduce from Lemma 4.8 that $|\hat{g}|$ is also of exponential type and hence we can apply the Paley-Wiener-Schwartz Theorem to $\hat{g}$ to deduce that g has compact support. Using results from the beginning of the chapter we can deduce that the only function belonging to $F_{[\lambda,\mathscr{S}']}$ with compact support is the zero function, so $g = 0$, which would imply that $h = 0$. Hence, if h has compact support, it must be the zero function. This completes the proof.

The hypotheses of Theorem 3.2 in Chapter 3 are that $\hat{g}\xi_\lambda$ is of exponential type, rather than supposing that h has compact support. (The Paley-Wiener Theorem for L^2 gives sufficient, not necessary conditions for h to have compact support.) We improve on this result using Theorem 4.4. The Paley-Wiener-Schwartz Theorem, which is an if and only if result, is used to prove that the relaxation spectrum h does not have compact support (unless it is the zero function) for $g \in F_{[\lambda,\mathscr{S}']}$. However, we introduce an important corollary below, which proves that h does not have compact support for $g \in F_{[\lambda,p]}$. Proving that h does not have compact support for $g \in F_{[\lambda,p]}$ is an important result, since this is the space of functions that Loy, Davies, Anderssen and Newbury [50] and [51] and Dodd [23] all work with in their calculations.

Corollary 4.5. *If* $h \in L^q(\mathbb{R})$ *has compact support, where* $h = \mathscr{F}^{-1}[\hat{g}(r) \cdot \cosh(\lambda r)]$ *for* $g \in F_{[\lambda,p]}$ *and* $\hat{g}(r) \cdot \cosh(\lambda r) \in L^p(\mathbb{R})$*, then* $h = 0$*.*

Since $L^q(\mathbb{R})$ is a subspace of $\mathscr{S}'(\mathbb{R})$, we can deduce that if h belongs to $L^q(\mathbb{R})$ (it also belongs to $\mathscr{S}'(\mathbb{R})$) and has compact support then it must be the zero function, using the Paley-Wiener-Schwartz Theorem and Theorem 4.4 and Lemma 4.7.

This result leads us to the question: what functions are in the space $F_{[\lambda,p]}$ and what properties can we deduce about these functions? We address this in the following section.

4.4 Strictly Positive Definite Functions

In this section we deduce further properties of the relaxation spectrum if additional hypotheses are satisfied.

We follow Champeney [17] and Cooper [19] in our definiton of a positive definite function:

Definition 4.9. *A real or complex valued function $f = f(x)$ is positive definite if:*

- *it is defined, bounded and continuous on $(-\infty, \infty)$, and*
- *at all x, $\bar{f}(-x) = f(x)$, (this implies an even function for real functions) and*
- *for any points $x_1, x_2, \ldots, x_N$, $(N = 1, 2, 3 \ldots)$ and any (complex) numbers $a_1, a_2, \ldots, a_N$*

$$\sum_{m=1}^{N} \sum_{n=1}^{N} f(x_m - x_n)\, a_m \bar{a}_n \geq 0. \tag{4.25}$$

For strictly positive definite we have equality in equation (4.25) only when $\underline{a} = (a_1, \ldots, a_N) = 0$.

We introduce an important result from Chang [18], a similar result can also be found in Fasshauer [26]:

Theorem 4.10. *Let μ be a nonzero, finite, Borel measure on $\mathbb{R}$ such that the carrier of μ is not a discrete set. Then the generalised Fourier transform $\hat{\mu}$ of μ is strictly positive definite on $\mathbb{R}$.*

We note that Theorem 4.10 is the converse of Bochner's theorem, that is, every (strictly) positive definite function F is the Fourier transform of a positive finite Borel measure. We deduce that Theorem 4.10 can be interpreted as an if and only if result.

We have the following result, for non-zero $g \in F_{[\lambda,p]}$:

Proposition 4.11. *For a* $p \in (1\ ,\ 2]$, *let non-zero* $g \in F_{[\lambda,p]}$. *Suppose that:*

$$\mu(B) = \int_B \hat{g}(x) \cosh(\lambda x)\ \mathrm{d}x \tag{4.26}$$

defines a finite Borel measure on $\mathbb{R}$. *Then* $h = \mathscr{F}^{-1}[\hat{g}\xi_\lambda]$ *is strictly positive definite.*

We note that the hypotheses of Proposition 4.11 are satisfied by some non-trivial elements of the space $F_{[\lambda,p]}$. The examples we introduce in section 4.5 satisfy Proposition 4.11.

Proof of Proposition 4.11

The support of $\hat{g} \cdot \cosh$ as a function is the same as the support of $\hat{g} \cdot \cosh$ as a measure density, which is just the support of $\hat{g}$. Hence, for our function, the carrier of $\hat{g} \cdot \cosh$ is the support of $\hat{g}$. The function $\hat{g}(x) \cdot \cosh(\lambda x)$, evaluated at $x = x_0$ is nonzero (as g is non-trivial); $\hat{g}(x) \cdot \cosh(\lambda x)$ is nonzero for $x \in (x_0 - \epsilon, x_0 + \epsilon)$ (by continuity of $\hat{g}\xi_\lambda$). Thus the support of $\hat{g} \cdot \cosh$ is not discrete. We can see that μ satisfies Theorem 4.10 and noting that the arguments in Theorem 4.10 are equally applicable to the inverse Fourier transform, we can deduce that $h = \mathscr{F}^{-1}[\hat{g} \cdot \xi_\lambda]$ is strictly positive definite.

The archetypal positive definite function is a Gaussian; the Gaussian functions are obvious candidates for our space $F_{[\lambda,p]}$. There is a sense in which being positive definite implies a decay condition. We see in Chang [18], that we can normalise a strictly positive definite function f such that $|f(x)| \leq 1$ and $f(0) = 1$, where the function decays both sides of the origin. We will consider the types of functions in the space $F_{[\lambda,p]}$ in the next section.

4.5 Functions in the space $F_{[\lambda,p]}$

Now, $\cosh(\lambda x)$ diverges rapidly to infinity as $|x| \to \infty$. Hence, we can deduce immediately that the function $\hat{g}$ must crush down significantly on the cosh function to ensure that the product of the two belongs to $L^p(\mathbb{R})$.

We note also that for $g \in F_{[\lambda,p]}$, g is an analytic function, and hence cannot have compact support.

We recall some basic facts about Fourier transforms (see, for example, Champeney [17]):

Theorem 4.12. *Suppose $f \in L^1(-\infty,\infty)$, then the Fourier transform $\hat{f}(y)$ of $f(x)$ is:*

- *bounded on $(-\infty,\infty)$, and at each y,*

$$\left|\hat{f}(y)\right| \leq \int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x, \tag{4.27}$$

- *everywhere continuous and indeed uniformly continuous on $(-\infty,\infty)$,*
- *$\hat{f}(y)$ tends to zero as $y \to \pm\infty$ (Riemann-Lebesgue Lemma).*

Since g is defined as belonging to $L^1(\mathbb{R})$ it follows that $\hat{g}$ is bounded on the whole of the real line, is uniformly continuous on the real line and $\hat{g}(y)$ tends to zero as $y \to \pm\infty$, for more details see Chapter 8 of Champeney [17].

In considering what functions belong to $F_{[\lambda,p]}$, we will consider functions $\hat{g}$ whose product with cosh belongs to $L^p(\mathbb{R})$. Then we will take inverse Fourier transforms and check that g belongs to $L^1(\mathbb{R})$. If the function $\hat{g}$ multiplied by $\cosh(\lambda x) = \left(e^{\lambda x} + e^{-\lambda x}\right)/2$ does in fact belong to $L^p(\mathbb{R})$, then the function $\hat{g}$ must have exponential decay at a rate greater than $e^{-\lambda x}$ for positive x and at least $e^{\lambda x}$ for negative x. An obvious choice for $\hat{g}$ is a Gaussian function, which decays rapidly as $|x| \to \infty$ and has the property of being less than or equal to one. Furthermore, when we take the inverse Fourier transform of a Gaussian, we get another Gaussian, which belongs to $L^1(\mathbb{R})$, that is, the Gaussian functions belong to the space $F_{[\lambda,p]}$.

Can we find other functions $\hat{g}$, that decay rapidly enough such that $\hat{g}\cosh(\lambda x) \in L^p(\mathbb{R})$?

The function $\hat{g} = \alpha/x^{\beta}$, where α and β are real numbers, decays rapidly as $|x| \to \infty$, however it will never beat $\cosh(\lambda x)$ as $|x| \to \infty$. The function $\hat{g}$ must have exponential decay at $\pm\infty$ in order to dominate $\cosh(\lambda x) = \left(e^{\lambda x} + e^{-\lambda x}\right)/2$, however, we might be able to find a function $\hat{g}$ that has exponential decay outside some interval $[-c, c]$ and takes some other form on $[-c, c]$.

4.5.1 Gaussian Functions

We begin by considering a Gaussian function: $B_1(x) = e^{-\alpha^2 x^2}$, where α is a positive real number.

Now, if we consider the product of $B_1(x)$ with cosh, it is clear that the exponential decay will crush down on the cosh such that the product will belong to $L^p(\mathbb{R})$.

We know that the Fourier transform of a Gaussian is another Gaussian, hence, it is obvious that $b_1 = \mathscr{F}^{-1}(B_1)$ will be a Gaussian function and hence belongs to $L^1(\mathbb{R})$.

A straightforward calculation gives that $b_1(t) = \sqrt{\pi}/2 \exp\{-t^2/4\alpha^2\}$.

It follows that $b_1 \in F_{[\lambda,p]}$. We deduce that the Gaussian functions belong to the space $F_{[\lambda,p]}$.

4.5.2 Gaussian Decay at Extremes

Now, we consider a slightly different function, we keep the Gaussian decay at the extremes and have some other function in a small interval. Consider

$$B_2(x) = \begin{cases} e^{-\epsilon^2 x^2}, & \text{if } -\infty < x \leq -c \\ e^{-c\epsilon^2 |x|}, & \text{if } -c < x < c \\ e^{-\epsilon^2 x^2}, & \text{if } c \leq x < \infty. \end{cases}$$

where ϵ and c are positive real numbers. $B_2(x)$ has the following graph:

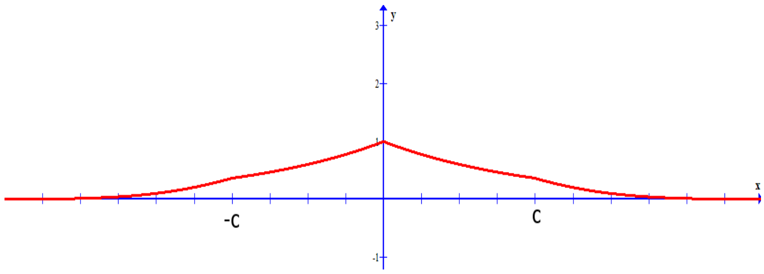

Figure 4.2: Gaussian decay at extremes and exponential decay locally $\epsilon = 0.25$ and $c = 2$

As we have mentioned above, our function $\hat{g}$ must have exponential decay (or be zero) at extremes to ensure that the product $\hat{g}\xi_\lambda \in L^2(\mathbb{R})$. We note that our choice of $\hat{g}$ on some closed interval could result in $g \notin L^1(\mathbb{R})$. For example, an obvious choice for $\hat{g}$ on some closed interval would be a constant, however, we note that when we take the inverse Fourier transform of a constant on a closed interval (difference of Heaviside functions) we obtain a function of the form $\sin(x)/x$ which is not in $L^1(\mathbb{R})$. Even when $\hat{g}$ has Gaussian decay at extremes, there are corresponding functions g that are not in the space $F_{[\lambda,p]}$. We demonstrate that the inverse Fourier transform, b_2 of $B_2(x)$ belongs to $L^1(\mathbb{R})$.

The inverse Fourier transform of $B_2(x)$ can be written as:

$$\begin{aligned}
\mathscr{F}^{-1}[B_2(x)] &= \int_{-\infty}^{\infty} B_2(x)\, e^{iyx}\,\mathrm{d}x \\
&= \int_{-\infty}^{-c} e^{-\epsilon^2 x^2} e^{iyx}\,\mathrm{d}x + \int_{-c}^{c} e^{-c\epsilon^2|x|} e^{iyx}\,\mathrm{d}x + \int_{c}^{\infty} e^{-\epsilon^2 x^2} e^{iyx}\,\mathrm{d}x \\
&= I_1 + I_2 + I_3. \qquad (4.28)
\end{aligned}$$

We work with the first and third integrals in the equation above, I_1 and I_3.

Consider

$$\begin{aligned} I_1 + I_3 &= \int_{-\infty}^{-c} e^{-\epsilon^2 x^2} e^{iyx}\,\mathrm{d}x + \int_{c}^{\infty} e^{-\epsilon^2 x^2} e^{iyx}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} e^{-\epsilon^2 x^2} e^{iyx}\,\mathrm{d}x - \int_{-c}^{c} e^{-\epsilon^2 x^2} e^{iyx}\,\mathrm{d}x, \end{aligned} \tag{4.29}$$

using properties of the pdf of a normal distribution, we have that:

$$\int_{-\infty}^{\infty} e^{-\epsilon^2 x^2} e^{iyx}\,\mathrm{d}x = \frac{\sqrt{\pi}}{\epsilon} \exp\left\{\frac{-y^2}{4\epsilon^2}\right\}. \tag{4.30}$$

Finally, we consider

$$\int_{-c}^{c} e^{-\epsilon^2 x^2} e^{iyx}\,\mathrm{d}x = \exp\left\{\frac{-y^2}{4\epsilon^2}\right\} \int_{-c}^{c} \exp\left\{-\epsilon^2 \left(x - \frac{iy}{2\epsilon^2}\right)^2\right\} \mathrm{d}x. \tag{4.31}$$

We make a change of variables: let $u = x - iy/2\epsilon^2$ such that $(\mathrm{d}u/\mathrm{d}x) = 1$, then equation (4.31) becomes:

$$\int_{-c}^{c} e^{-\epsilon^2 x^2} e^{iyx}\,\mathrm{d}x = \exp\left\{\frac{-y^2}{4\epsilon^2}\right\} \int_{-c}^{c} e^{-\epsilon^2 u^2}\,\mathrm{d}u. \tag{4.32}$$

The definition of the erf function is as follows:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,\mathrm{d}t. \tag{4.33}$$

Since $e^{-\epsilon^2 u^2}$ is an even function it follows that:

$$\exp\left\{\frac{-y^2}{4\epsilon^2}\right\}\int_{-c}^{c} e^{-\epsilon^2 u^2}\,\mathrm{d}u = 2\exp\left\{\frac{-y^2}{4\epsilon^2}\right\}\int_{0}^{c} e^{-\epsilon^2 u^2}\,\mathrm{d}u. \tag{4.34}$$

We can use equation (4.32) to express equation (4.34) as:

$$\exp\left\{\frac{-y^2}{4\epsilon^2}\right\}\int_{-c}^{c} e^{-\epsilon^2 u^2}\,\mathrm{d}u = 2\exp\left\{\frac{-y^2}{4\epsilon^2}\right\}\frac{\sqrt{\pi}\operatorname{erf}(c\epsilon)}{2\epsilon}. \tag{4.35}$$

Using the results of (4.30) and (4.34), we can express equation (4.35) as:

$$\begin{aligned} I_1 + I_3 &= \frac{\sqrt{\pi}}{\alpha}\exp\left\{\frac{-y^2}{4\epsilon^2}\right\} - \exp\left\{\frac{-y^2}{4\epsilon}\right\}\frac{\sqrt{\pi}\operatorname{erf}(c\epsilon)}{\epsilon} \\ &= \left[1 - \operatorname{erf}(c\epsilon)\right]\frac{\sqrt{\pi}}{\epsilon}\exp\left\{\frac{-y^2}{4\epsilon}\right\}. \end{aligned} \tag{4.36}$$

Now, $[1 - \operatorname{erf}(c\epsilon)]$ is a constant between zero and 2, and so it follows that $I_1 + I_3$ belongs to $L^1(\mathbb{R})$.

Now, we need only to calculate I_2 and show that it belongs to $L^1(\mathbb{R})$.

$$\begin{aligned} I_2 &= \int_{-c}^{c} e^{-c\epsilon^2|x|} e^{iyx}\,\mathrm{d}x \\ &= \int_{0}^{c} e^{-c\epsilon^2 x+iyx}\,\mathrm{d}x + \int_{-c}^{0} e^{c\epsilon^2 x+iyx}\,\mathrm{d}x \\ &= \left[\frac{e^{\left(iy-c\epsilon^2\right)x}}{iy-c\epsilon^2}\right]_0^c + \left[\frac{e^{\left(iy+c\epsilon^2\right)x}}{iy+c\epsilon^2}\right]_{-c}^0 \\ &= \left[\frac{e^{\left(icy-c^2\epsilon^2\right)}-1}{iy-c\epsilon^2}\right] + \left[\frac{1-e^{\left(-icy-c^2\epsilon^2\right)}}{iy+c\epsilon^2}\right] \\ &= \frac{2c\epsilon^2+2e^{-c^2\epsilon^2}\left(y\sin(cy)-c\epsilon^2\cos(cy)\right)}{(y^2+c^2\epsilon^4)}, \end{aligned} \tag{4.37}$$

which has the following graph:

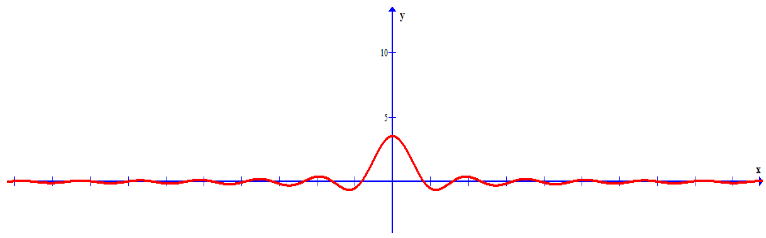

Figure 4.3: Graph of I_2

that is, I_2 belongs to $L^1(\mathbb{R})$. We write $\mathscr{F}^{-1}[B_2(x)] = b_2(y)$, where

$$\begin{aligned} b_2(y) &= I_3+I_2+I_3 = [1-\mathrm{erf}(c\epsilon)]\frac{\sqrt{\pi}}{\epsilon}\exp\left\{\frac{-y^2}{4\epsilon}\right\} \\ &+ \frac{2c\epsilon^2+2e^{-c^2\epsilon^2}\left(y\sin(cy)-c\epsilon^2\cos(cy)\right)}{(y^2+c^2\epsilon^4)} \end{aligned} \tag{4.38}$$

and if follows that $b_2 \in F_{[\lambda,p]}$.

4.5.3 Sech Functions

Another possible function that could belong to the space $F_{[\lambda,p]}$ is the sech function.

Define the following function: $B_3(x) = \operatorname{sech}(\alpha x)$, where $\alpha > \lambda$.

We consider the product $B_3(x) \cdot \cosh(\lambda x)$:

$$\begin{aligned} B_3(x) \cdot \cosh(\lambda x) &= \operatorname{sech}(\alpha x) \cosh(\lambda x) \\ &= \frac{2}{e^{\alpha x} + e^{-\alpha x}} \frac{e^{\lambda x} + e^{-\lambda x}}{2} \\ &= \frac{e^{\lambda x} + e^{-\lambda x}}{e^{\alpha x} + e^{-\alpha x}}, \end{aligned} \tag{4.39}$$

which has the following graph:

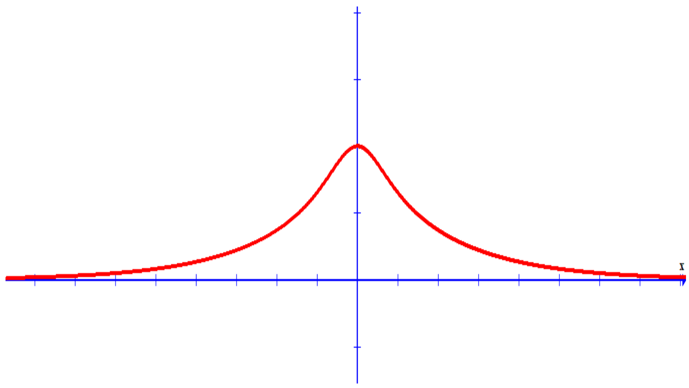

Figure 4.4: Graph of $B_3(x) \cdot \cosh(\lambda x)$

which clearly belongs to $L^p(\mathbb{R})$.

We need only to show that the inverse Fourier transform of B_3 belongs to $L^1(\mathbb{R})$.

We have already demonstrated in Chapter 2 and in Fourier Transform 1 in the Appendix that the Fourier transform of a sech function is another sech function, hence we already know that the inverse Fourier transform of $B_3(x)$ is a sech function which belongs to $L^1(\mathbb{R})$. We write $\mathscr{F}^{-1}[B_3(x)] = b_3(y)$, and if follows that $b_3 \in F_{[\lambda,p]}$. Hence, the sech functions are in the space $F_{[\lambda,p]}$.

4.5.4 Functions with Compact Support

Although we have emphasised that g cannot have compact support, the Fourier transform of g could have compact support. We demonstrate with an example.

Suppose we have that $\hat{g}$ is the following function:

$$B_4(x) = \begin{cases} \beta^2 - x^2, & \text{if } -\beta \leq x \leq \beta \\ 0, & \text{otherwise.} \end{cases}$$

If we consider the product of $B_4(x)$ with $\cosh(\lambda x)$ we can deduce immediately that the product will belong to $L^p(\mathbb{R})$.

We calculate the inverse Fourier transform of $B_4(x)$ and demonstrate that it belongs to $L^1(\mathbb{R})$ and does not have compact support.

$$\begin{aligned}
\mathscr{F}^{-1}\left[B_4(x)\right] &= \int_{-\infty}^{\infty} B_4(x)\, e^{ixy}\, \mathrm{d}x \\
&= \int_{-\beta}^{\beta} \left(\beta^2 - x^2\right) e^{ixy}\, \mathrm{d}x \\
&= \beta^2 \int_{-\beta}^{\beta} e^{ixy}\, \mathrm{d}x - \int_{-\beta}^{\beta} x^2 e^{ixy}\, \mathrm{d}x \\
&= \frac{\beta^2}{iy}\left[e^{\beta iy} - e^{-\beta iy}\right] - I_1.
\end{aligned} \tag{4.40}$$

We use integration by parts to solve for I_1:

$$\begin{aligned}
I_1 = \int_{-\beta}^{\beta} x^2 e^{ixy}\, \mathrm{d}x &= \left[\frac{x^2 e^{ixy}}{iy}\right]_{-\beta}^{\beta} - \int_{-\beta}^{\beta} \frac{2x e^{ixy}}{iy}\, \mathrm{d}x \\
&= \frac{\beta^2}{iy}\left[e^{\beta iy} - e^{-\beta iy}\right] - \frac{2}{iy}\int_{-\beta}^{\beta} x e^{ixy}\, \mathrm{d}x \\
&= \frac{\beta^2}{iy}\left[e^{\beta iy} - e^{-\beta iy}\right] - \frac{2}{iy} I_2.
\end{aligned} \tag{4.41}$$

We use integration by part to solve for I_2:

$$\begin{aligned}
I_2 = \int_{-\beta}^{\beta} x e^{ixy}\, \mathrm{d}x &= \left[\frac{x e^{ixy}}{iy}\right]_{-\beta}^{\beta} - \int_{-\beta}^{\beta} \frac{e^{ixy}}{iy}\, \mathrm{d}x \\
&= \frac{\beta}{iy}\left[e^{\beta iy} + e^{-\beta iy}\right] - \left[\frac{e^{ixy}}{(iy)^2}\right]_{-\beta}^{\beta} \\
&= \frac{\beta}{iy}\left[e^{\beta iy} + e^{-\beta iy}\right] + \frac{1}{y^2}\left[e^{\beta iy} - e^{-\beta iy}\right].
\end{aligned} \tag{4.42}$$

Combining the above, we have that:

$$\begin{aligned}
\mathscr{F}^{-1}[B_4(x)] &= \frac{\beta^2}{iy}\left[e^{\beta iy}-e^{-\beta iy}\right]-\frac{\beta^2}{iy}\left[e^{\beta iy}-e^{-\beta iy}\right]\\
&+ \frac{2}{iy}\frac{\beta}{iy}\left[e^{\beta iy}+e^{-\beta iy}\right]+\frac{2}{iy}\frac{1}{y^2}\left[e^{\beta iy}-e^{-\beta iy}\right]\\
&= \frac{2}{iy}\left\{\frac{\beta}{iy}\left[e^{\beta iy}+e^{-\beta iy}\right]+\frac{1}{y^2}\left[e^{\beta iy}-e^{-\beta iy}\right]\right\}\\
&= \frac{2}{iy}\left\{\frac{\beta}{iy}\left[2\cos(\beta y)\right]+\frac{1}{y^2}\left[2i\sin(\beta y)\right]\right\}\\
&= -\frac{4\beta\cos(\beta y)}{y^2}+\frac{4i\sin(\beta y)}{iy^3}\\
&= \frac{4\sin(\beta y)}{y^3}-\frac{4\beta\cos(\beta y)}{y^2}, \qquad (4.43)
\end{aligned}$$

which has the following graph:

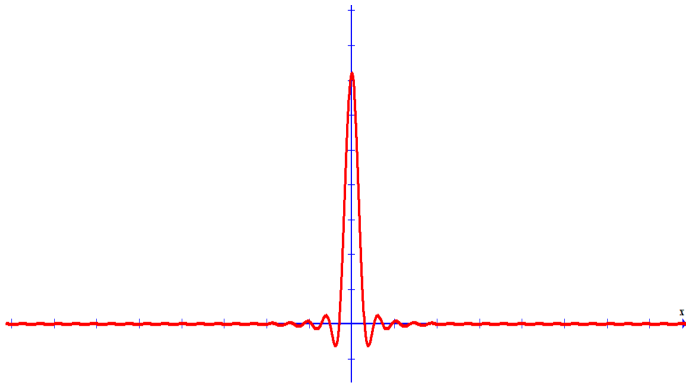

Figure 4.5: Graph of inverse Fourier transform of $B4(x)$ with $\beta = 3$

which belongs to $L^1(\mathbb{R})$ and is not compactly supported. We write $\mathscr{F}^{-1}[B_4(x)] = b_4(y)$, and if follows that $b_4 \in F_{[\lambda,p]}$.

4.6 Summary

In this chapter we have proven for all $g \in F_{[\lambda,\mathscr{S}']}$ that if $|\xi_\lambda \cdot \hat{g}|$ has a bound of exponential type then $|\hat{g}|$ has a bound of exponential type. Using the Paley-Wiener-Schwartz Theorem, we have also proven that for all $g \in F_{[\lambda,\mathscr{S}']}$, the relaxation spectrum; h, defined as $h = \mathscr{F}^{-1}[\xi_\lambda \cdot \hat{g}]$, does not have compact support. Furthermore, we note as a corollary that, since L^p is a subspace of $\mathscr{S}'$, the relaxation spectrum does not have compact support for all $g \in F_{[\lambda,p]}$, unless $h = 0$. This is a very important result since the result of compact support in the work of Loy, Davies and Anderssen [51] works with functions g in the space $F_{[\lambda,p]}$.

We have also considered what further properties may be deduced about the function $g \in F_{[\lambda,p]}$ and the relaxation spectrum h. We have demonstrated that if $\mu(B)$, as defined in equation (4.26), defines a finite Borel measure on $\mathbb{R}$, then the relaxation spectrum h is strictly positive definite. We note that the Gaussian functions are a prime example of a strictly positive definite function and as we have already mentioned above, the Gaussian functions are an obvious candidate for our space $F_{[\lambda,p]}$.

Following this we have considered the types of functions that are in the space $F_{[\lambda,p]} = \{g \in L^1(\mathbb{R}) \mid \xi_\lambda \cdot \hat{g} \in L^p(\mathbb{R})\}$. What we have found is that the function $\hat{g}$ must have exponential decay (or be zero) outside some interval $[a, b]$, where a and b are real numbers, for the product $\xi_\lambda \cdot \hat{g}$ to belong to $L^p(\mathbb{R})$. In the next chapter we will make use of these functions again to calculate their corresponding relaxation spectra. We deduce some interesting results. Although the relaxation spectrum does not have compact support, for a subclass in the spaces of functions we are dealing with, it does become insignificant outside a closed interval. We define this concept as "compact essential numerical support".

Chapter 5

Weak Notion of Support for the Relaxation Spectrum

5.1 Compact Essential Numerical Support

We have demonstrated in the previous two chapters that the task of recovering information about the support of the relaxation spectrum is indeed difficult. In fact we have demonstrated that for many spaces of functions, the relaxation spectrum does not have compact support and hence sampling localisation does not have a solid theoretical base. Sampling localisation has however proved to be useful. Can we find a result that could be of use to the practical rheologist?

In the previous chapter, in order to gain an understanding of the types of functions we are dealing with, we looked at possible functions that belong to the space $F_{[\lambda,p]}$. In this chapter we calculate the relaxation spectra corresponding to these functions. We deduce from the graphs of these spectra that they become insignificant outside a closed interval. We denote this as *compact essential numerical support*.

We make this concept more rigorous by considering possible options for calculating an interval of compact essential numerical support for the relaxation spectrum. We note how different definitions of compact essential numerical support can allow us to measure different aspects of the decay.

We demonstrate using results of Faddeeva functions that for a subspace of $F_{[\lambda,p]}$, (those functions with Gaussian decay at far-field), the relaxation spectrum has supremum norm compact essential numerical support. We do this by splitting $\hat{g}$ into two parts; the part with Gaussian decay at far-field and the near-field part. We make estimates on these two parts separately. We evaluate the accuracy of our method by applying it to one of the examples that we calculate at the beginning of this chapter. The estimate is relatively close to the true interval of compact essential numerical support. However we note that there may be other methods and estimates that need to be considered.

5.2 Calculating Relaxation Spectra

We demonstrated in Chapter 4 that the Gaussian and sech functions along with the functions defined as b_2 and b_4 belong to $F_{[\lambda,p]}$. We will now calculate their corresponding relaxation spectra.

We begin with a Gaussian function; in Chapter 4 we defined $b_1(y) = \sqrt{\pi}/\alpha \exp\{-y^2/4\alpha^2\}$, which belongs to $F_{[\lambda,p]}$. The function $b_1(x)$ has Fourier transform $B_1(x) = e^{-\alpha^2 y^2}$.

We calculate:

$$
\begin{aligned}
h_1(t) &= \mathscr{F}^{-1}\left[B_1(x)\cdot\cosh(\lambda x)\right] \\
&= \int_{-\infty}^{\infty} e^{-\alpha^2x^2}\left(\frac{e^{\lambda x}+e^{-\lambda x}}{2}\right)e^{ixt}\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\exp\left\{-\alpha^2x^2+\lambda x+ixt\right\}+\exp\left\{-\alpha^2x^2-\lambda x+ixt\right\}\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\exp\left\{-\alpha^2\left(x^2-\frac{(\lambda+it)\,x}{\alpha}\right)\right\}+\exp\left\{-\alpha^2\left(x^2+\frac{(it-\lambda)\,x}{\alpha}\right)\right\}\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\exp\left\{-\alpha^2\left(x-\frac{(\lambda+it)}{2\alpha^2}\right)^2\right\}\exp\left\{\frac{(\lambda+it)^2}{4\alpha^2}\right\}\,\mathrm{d}x \\
&+ \frac{1}{2}\int_{-\infty}^{\infty}\exp\left\{-\alpha^2\left(x-\frac{(it-\lambda)}{2\alpha^2}\right)^2\right\}\exp\left\{\frac{(it-\lambda)^2}{4\alpha^2}\right\}\,\mathrm{d}x \\
&= \frac{1}{2}\exp\left\{\frac{(\lambda+it)^2}{4\alpha^2}\right\}\int_{-\infty}^{\infty}\exp\left\{-\alpha^2\left(x-\frac{(\lambda+it)}{2\alpha^2}\right)^2\right\}\,\mathrm{d}x \\
&+ \frac{1}{2}\exp\left\{\frac{(it-\lambda)^2}{4\alpha^2}\right\}\int_{-\infty}^{\infty}\exp\left\{-\alpha^2\left(x-\frac{(it-\lambda)}{2\alpha^2}\right)^2\right\}\,\mathrm{d}x. \qquad (5.1)
\end{aligned}
$$

Making use of the pdf of a Normal distribution we reduce this to:

$$
\begin{aligned}
h_1(t) &= \frac{1}{2}\exp\left\{\frac{(\lambda+it)^2}{4\alpha^2}\right\}\frac{\sqrt{\pi}}{\alpha}+\frac{1}{2}\exp\left\{\frac{(it-\lambda)^2}{4\alpha^2}\right\}\frac{\sqrt{\pi}}{\alpha} \\
&= \frac{\sqrt{\pi}}{2\alpha}\left[\exp\left\{\frac{\lambda^2+2i\lambda t-t^2}{4\alpha^2}\right\}+\exp\left\{\frac{-t^2-2i\lambda t+\lambda^2}{4\alpha^2}\right\}\right] \\
&= \frac{\sqrt{\pi}}{2\alpha}e^{\lambda^2/4\alpha^2}e^{-t^2/4\alpha^2}\left[\exp\left\{\frac{i\lambda t}{2\alpha^2}\right\}+\exp\left\{\frac{-i\lambda t}{2\alpha^2}\right\}\right] \\
&= \frac{\sqrt{\pi}}{2\alpha}e^{\lambda^2/4\alpha^2}e^{-t^2/4\alpha^2}\left[2\cos\left(\frac{\lambda t}{2\alpha^2}\right)\right] \\
&= \frac{2\sqrt{\pi}}{2\alpha}e^{\lambda^2/4\alpha^2}\exp\left\{\frac{-t^2}{4\alpha^2}\right\}\left[\cos\left(\frac{\lambda t}{2\alpha^2}\right)\right], \qquad (5.2)
\end{aligned}
$$

which has the following graph:

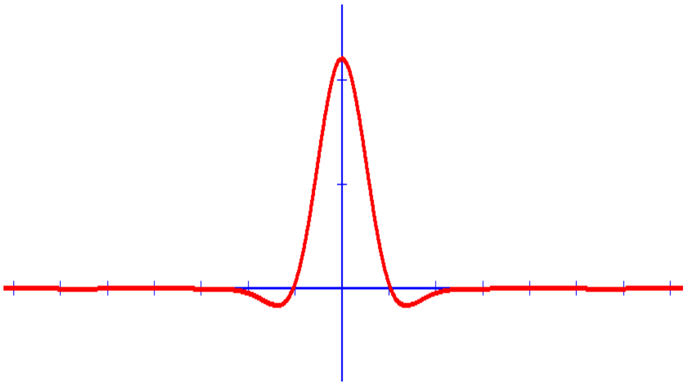

Figure 5.1: Graph of h_1 with $\alpha = 1$ and $\lambda = 1.5$

Clearly, h_1 has compact essential numerical support, that is, it becomes almost insignificant outside some closed interval. For the exact definition of compact essential numerical support, we refer the reader to Definition 5.1.

Next we consider the function $b_2(y)$ that we calculated in the previous chapter, which belongs to $F_{[\lambda,p]}$. The Fourier transform of $b_2(y)$ is $B_2(x)$ defined as:

$$B_2(x) = \begin{cases} e^{-\epsilon^2 x^2}, & \text{if } -\infty < x \leq -c \\ e^{-c\epsilon^2 |x|}, & \text{if } -c < x < c \\ e^{-\epsilon^2 x^2}, & \text{if } c \leq x < \infty \end{cases}$$

for ϵ and c positive real numbers.

We calculate $h_2(t)$:

$$\begin{aligned}
h_2(t) &= \mathscr{F}^{-1}\left[B_2(x)\cdot\cosh(\lambda x)\right] \\
&= \int_{-\infty}^{-c} e^{-\epsilon^2x^2}\left(\frac{e^{\lambda x}+e^{-\lambda x}}{2}\right)e^{ixt}\,\mathrm{d}x + \int_{-c}^{c} e^{-c\epsilon^2|x|}\left(\frac{e^{\lambda x}+e^{-\lambda x}}{2}\right)e^{ixt}\,\mathrm{d}x \\
&+ \int_{c}^{\infty} e^{-\epsilon^2x^2}\left(\frac{e^{\lambda x}+e^{-\lambda x}}{2}\right)e^{ixt}\,\mathrm{d}x \\
&= C_1+C_2+C_3. \qquad (5.3)
\end{aligned}$$

We begin by noting that:

$$\begin{aligned}
C_1+C_3 &= \int_{-\infty}^{-c} e^{-\epsilon^2x^2}\left(\frac{e^{\lambda x}+e^{-\lambda x}}{2}\right)e^{ixt}\,\mathrm{d}x + \int_{c}^{\infty} e^{-\epsilon^2x^2}\left(\frac{e^{\lambda x}+e^{-\lambda x}}{2}\right)e^{ixt}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} e^{-\epsilon^2x^2}\left(\frac{e^{\lambda x}+e^{-\lambda x}}{2}\right)e^{ixt}\,\mathrm{d}x - \int_{-c}^{c} e^{-\epsilon^2x^2}\left(\frac{e^{\lambda x}+e^{-\lambda x}}{2}\right)e^{ixt}\,\mathrm{d}x.
\end{aligned} \qquad (5.4)$$

Making use of the results above, (see equations (5.1) and (5.2)), we can express equation (5.4) as:

$$C_1+C_3 = \frac{2\sqrt{\pi}}{\epsilon}e^{\lambda^2/4\epsilon^2}e^{-t^2/4\epsilon^2}\cos\left(\frac{\lambda t}{2\epsilon^2}\right) - \int_{-c}^{c} e^{-\epsilon^2x^2}\left(\frac{e^{\lambda x}+e^{-\lambda x}}{2}\right)e^{ixt}\,\mathrm{d}x. \qquad (5.5)$$

We calculate the second integral in equation (5.5), using results from equations (5.1) and (5.2) :

$$
\begin{aligned}
I_\epsilon &= \int_{-c}^{c} e^{-\epsilon^2 x^2}\left(\frac{e^{\lambda x}+e^{-\lambda x}}{2}\right) e^{ixt}\,\mathrm{d}x \\
&= \frac{1}{2}\exp\left\{\frac{1}{4\epsilon^2}(\lambda+it)^2\right\}\int_{-c}^{c}\exp\left\{-\epsilon^2\left(x-\frac{1}{2\epsilon^2}(\lambda+it)\right)^2\right\}\mathrm{d}x \\
&+ \frac{1}{2}\exp\left\{\frac{1}{4\epsilon^2}(\lambda-it)^2\right\}\int_{-c}^{c}\exp\left\{-\epsilon^2\left(x+\frac{1}{2\epsilon^2}(\lambda-it)\right)^2\right\}\mathrm{d}x. \qquad (5.6)
\end{aligned}
$$

Now let $\theta^+ = \frac{1}{2\epsilon^2}(\lambda+it)$ and $\theta^- = \frac{1}{2\epsilon^2}(\lambda-it)$, then equation (5.6) becomes:

$$
\begin{aligned}
I_\epsilon &= \frac{1}{2}\exp\left\{\frac{1}{4\epsilon^2}(\lambda+it)^2\right\}\int_{-c}^{c}\exp\left\{-\epsilon^2\left(x-\theta^+\right)^2\right\}\mathrm{d}x \\
&+ \frac{1}{2}\exp\left\{\frac{1}{4\epsilon^2}(\lambda-it)^2\right\}\int_{-c}^{c}\exp\left\{-\epsilon^2\left(x+\theta^-\right)^2\right\}\mathrm{d}x. \qquad (5.7)
\end{aligned}
$$

Define $s = x-\theta^+$ then $\mathrm{d}s = \mathrm{d}x$ and $r = x+\theta^-$ then $\mathrm{d}r = \mathrm{d}x$. We write equation (5.7) as:

$$
\begin{aligned}
I_\epsilon &= \frac{1}{2}\exp\left\{\frac{1}{4\epsilon^2}(\lambda+it)^2\right\}\int_{-c-\theta^+}^{c-\theta^+}\exp\left\{-\epsilon^2 s^2\right\}\mathrm{d}s \\
&+ \frac{1}{2}\exp\left\{\frac{1}{4\epsilon^2}(\lambda-it)^2\right\}\int_{-c+\theta^-}^{c+\theta^-}\exp\left\{-\epsilon^2 r^2\right\}\mathrm{d}r \qquad (5.8)
\end{aligned}
$$

and noting that the anti-derivative of $e^{-a^2x^2}$ is $\left(\sqrt{\pi}\mathrm{erf}\,(ax)\right)/2a$, we can express equation (5.8) as:

$$
\begin{aligned}
I_\epsilon = \quad &= \frac{1}{2}\exp\left\{\frac{1}{4\epsilon^2}(\lambda+it)^2\right\}\left[\frac{\sqrt{\pi}\operatorname{erf}(\epsilon s)}{2\epsilon}\right]_{-c-\theta^+}^{c-\theta^+} \\
&+ \frac{1}{2}\exp\left\{\frac{1}{4\epsilon^2}(\lambda-it)^2\right\}\left[\frac{\sqrt{\pi}\operatorname{erf}(\epsilon r)}{2\epsilon}\right]_{-c+\theta^-}^{c+\theta^-} \\
&= \frac{\sqrt{\pi}}{4\epsilon}\exp\left\{\frac{1}{4\epsilon^2}(\lambda+it)^2\right\}[\operatorname{erf}(\epsilon s)]_{-c-\theta^+}^{c-\theta^+} \\
&+ \frac{\sqrt{\pi}}{4\epsilon}\exp\left\{\frac{1}{4\epsilon^2}(\lambda-it)^2\right\}[\operatorname{erf}(\epsilon r)]_{-c+\theta^-}^{c+\theta^-} \\
&= \frac{\sqrt{\pi}}{4\epsilon}e^{\lambda^2/4\epsilon^2}e^{i\lambda t/2\epsilon^2}e^{-t^2/4\epsilon^2}\,[\operatorname{erf}(\epsilon s)]_{-c-\theta^+}^{c-\theta^+} \\
&+ \frac{\sqrt{\pi}}{4\epsilon}e^{\lambda^2/4\epsilon^2}e^{-i\lambda t/2\epsilon^2}e^{-t^2/4\epsilon^2}\,[\operatorname{erf}(\epsilon r)]_{-c+\theta^-}^{c+\theta^-}\,.
\end{aligned}
\tag{5.9}
$$

We evaluate the two error functions above:

$$
\begin{aligned}
[\operatorname{erf}(\epsilon s)]_{-c-\theta^+}^{c-\theta^+} &= \left[\operatorname{erf}\left(\epsilon\left(c-\theta^+\right)\right)-\operatorname{erf}\left(\epsilon\left(-c-\theta^+\right)\right)\right] \\
&= \left[\operatorname{erf}\left(c\epsilon-\epsilon\theta^+\right)-\operatorname{erf}\left(-c\epsilon-\epsilon\theta^+\right)\right] \\
&= \left[\operatorname{erf}\left(c\epsilon-\frac{1}{2\epsilon}(\lambda+it)\right)-\operatorname{erf}\left(-c\epsilon-\frac{1}{2\epsilon}(\lambda+it)\right)\right] \\
&= \left[\operatorname{erf}\left(c\epsilon-\frac{\lambda}{2\epsilon}-\frac{it}{2\epsilon}\right)+\operatorname{erf}\left(c\epsilon+\frac{\lambda}{2\epsilon}+\frac{it}{2\epsilon}\right)\right], \qquad (5.10)
\end{aligned}
$$

$$\begin{aligned}
\left[\operatorname{erf}(\epsilon r)\right]_{-c+\theta^-}^{c+\theta^-} &= \left[\operatorname{erf}\left(\epsilon\left(c+\theta^-\right)\right) - \operatorname{erf}\left(\epsilon\left(-c+\theta^-\right)\right)\right] \\
&= \left[\operatorname{erf}\left(c\epsilon+\epsilon\theta^-\right) - \operatorname{erf}\left(-c\epsilon+\epsilon\theta^-\right)\right] \\
&= \left[\operatorname{erf}\left(c\epsilon+\frac{1}{2\epsilon}\left(\lambda-it\right)\right) - \operatorname{erf}\left(-c\epsilon+\frac{1}{2\epsilon}\left(\lambda-it\right)\right)\right] \\
&= \left[\operatorname{erf}\left(c\epsilon+\frac{\lambda}{2\epsilon}-\frac{it}{2\epsilon}\right) + \operatorname{erf}\left(c\epsilon-\frac{\lambda}{2\epsilon}+\frac{it}{2\epsilon}\right)\right]. \qquad (5.11)
\end{aligned}$$

Combining the above, it follows that

$$\begin{aligned}
I_\epsilon &= \frac{\sqrt{\pi}}{4\epsilon}e^{\lambda^2/4\epsilon^2}e^{i\lambda t/2\epsilon^2}e^{-t^2/4\epsilon^2}\left[\operatorname{erf}\left(c\epsilon-\frac{\lambda}{2\epsilon}-\frac{it}{2\epsilon}\right)+\operatorname{erf}\left(c\epsilon+\frac{\lambda}{2\epsilon}+\frac{it}{2\epsilon}\right)\right] \\
&+ \frac{\sqrt{\pi}}{4\epsilon}e^{\lambda^2/4\epsilon^2}e^{-i\lambda t/2\epsilon^2}e^{-t^2/4\epsilon^2}\left[\operatorname{erf}\left(c\epsilon+\frac{\lambda}{2\epsilon}-\frac{it}{2\epsilon}\right)+\operatorname{erf}\left(c\epsilon-\frac{\lambda}{2\epsilon}+\frac{it}{2\epsilon}\right)\right].
\end{aligned} \qquad (5.12)$$

Next, we introduce the Faddeeva function, for which we have tables of values (see Abramowitz and Stegun [2]), which is defined as:

$$w(z) = \exp\left\{-z^2\right\}\operatorname{erfc}(-iz) = \exp\left\{-z^2\right\}\left[1-\operatorname{erf}(-iz)\right] \qquad (5.13)$$

for $z \in \mathbb{C}$ where erf is the error function and erfc is the complementary error function.

We would like to express equation (5.12) as a sum of Faddeeva functions. We write I_ϵ as:

$$I_\epsilon = \frac{\sqrt{\pi}}{4\epsilon}\left[F_1 + F_2 + F_3 + F_4\right]$$

and consider each F_n in turn and demonstrate that we can write them as $e^{-z^2}\mathrm{erf}\left(-iz\right) = e^{-z^2} - w\left(z\right)$, where $w\left(z\right)$ is the Faddeeva function.

$$\begin{aligned}
F_1 &= \exp\left\{\frac{\lambda^2 + 2i\lambda t - t^2}{4\epsilon^2}\right\}\left[\mathrm{erf}\left(c\epsilon - \frac{\lambda}{2\epsilon} - \frac{it}{2\epsilon}\right)\right] \\
&= e^{i\epsilon t - c^2\epsilon^2 + \lambda\epsilon} e^{-i\epsilon t + c^2\epsilon^2 - \lambda\epsilon} \exp\left\{\frac{\lambda^2 + 2i\lambda t - t^2}{4\epsilon^2}\right\}\left[\mathrm{erf}\left(c\epsilon - \frac{\lambda}{2\epsilon} - \frac{it}{2\epsilon}\right)\right] \\
&= e^{i\epsilon t - c^2\epsilon^2 + \lambda\epsilon} \exp\left\{\frac{\lambda^2 + 2i\lambda t - t^2}{4\epsilon^2} - i\epsilon t + c^2\epsilon^2 - \lambda\epsilon\right\}\left[\mathrm{erf}\left(c\epsilon - \frac{\lambda}{2\epsilon} - \frac{it}{2\epsilon}\right)\right] \\
&= e^{i\epsilon t - c^2\epsilon^2 + \lambda\epsilon} \exp\left\{-\left(\frac{t}{2\epsilon} + i\left(c\epsilon - \frac{\lambda}{2\epsilon}\right)\right)^2\right\}\left[\mathrm{erf}\left(c\epsilon - \frac{\lambda}{2\epsilon} - \frac{it}{2\epsilon}\right)\right] \\
&= e^{i\epsilon t - c^2\epsilon^2 + \lambda\epsilon}\left[\exp\left\{-\left(\frac{t}{2\epsilon} + i\left(c\epsilon - \frac{\lambda}{2\epsilon}\right)\right)^2\right\} - w\left(\frac{t}{2\epsilon} + i\left(c\epsilon - \frac{\lambda}{2\epsilon}\right)\right)\right] \\
&= \exp\left\{\frac{\lambda^2 + 2i\lambda t - t^2}{4\epsilon^2}\right\} - \exp\left\{i\epsilon t - c^2\epsilon^2 + \lambda\epsilon\right\} w\left(\frac{t}{2\epsilon} + i\left(c\epsilon - \frac{\lambda}{2\epsilon}\right)\right). \qquad (5.14)
\end{aligned}$$

We perform similar calculations for F_2:

$$\begin{aligned}
F_2 &= \exp\left\{\frac{\lambda^2+2i\lambda t-t^2}{4\epsilon^2}\right\}\left[\mathrm{erf}\left(c\epsilon+\frac{\lambda}{2\epsilon}+\frac{it}{2\epsilon}\right)\right] \\
&= e^{-i\epsilon t-c^2\epsilon^2-\lambda\epsilon}e^{i\epsilon t+c^2\epsilon^2+\lambda\epsilon}\exp\left\{\frac{\lambda^2+2i\lambda t-t^2}{4\epsilon^2}\right\}\left[\mathrm{erf}\left(c\epsilon+\frac{\lambda}{2\epsilon}+\frac{it}{2\epsilon}\right)\right] \\
&= e^{-i\epsilon t-c^2\epsilon^2-\lambda\epsilon}\exp\left\{\frac{\lambda^2+2i\lambda t-t^2}{4\epsilon^2}+i\epsilon t+c^2\epsilon^2+\lambda\epsilon\right\}\left[\mathrm{erf}\left(c\epsilon+\frac{\lambda}{2\epsilon}+\frac{it}{2\epsilon}\right)\right] \\
&= e^{-i\epsilon t-c^2\epsilon^2-\lambda\epsilon}\exp\left\{-\left(-\frac{t}{2\epsilon}+i\left(c\epsilon+\frac{\lambda}{2\epsilon}\right)\right)^2\right\}\left[\mathrm{erf}\left(c\epsilon+\frac{\lambda}{2\epsilon}+\frac{it}{2\epsilon}\right)\right] \\
&= e^{-i\epsilon t-c^2\epsilon^2-\lambda\epsilon}\left[\exp\left\{-\left(-\frac{t}{2\epsilon}+i\left(c\epsilon+\frac{\lambda}{2\epsilon}\right)\right)^2\right\}-w\left(-\frac{t}{2\epsilon}+i\left(c\epsilon+\frac{\lambda}{2\epsilon}\right)\right)\right] \\
&= \exp\left\{\frac{\lambda^2+2i\lambda t-t^2}{4\epsilon^2}\right\}-\exp\left\{-i\epsilon t-c^2\epsilon^2-\lambda\epsilon\right\}w\left(-\frac{t}{2\epsilon}+i\left(c\epsilon+\frac{\lambda}{2\epsilon}\right)\right)
\end{aligned} \tag{5.15}$$

For F_3:

$$\begin{aligned}
F_3 &= \exp\left\{\frac{\lambda^2-2i\lambda t-t^2}{4\epsilon^2}\right\}\left[\mathrm{erf}\left(c\epsilon+\frac{\lambda}{2\epsilon}-\frac{it}{2\epsilon}\right)\right] \\
&= e^{i\epsilon t-c^2\epsilon-\lambda\epsilon}e^{-i\epsilon t+c^2\epsilon^2+\lambda\epsilon}\exp\left\{\frac{\lambda^2-2i\lambda t-t^2}{4\epsilon^2}\right\}\left[\mathrm{erf}\left(c\epsilon+\frac{\lambda}{2\epsilon}-\frac{it}{2\epsilon}\right)\right] \\
&= e^{i\epsilon t-c^2\epsilon^2-\lambda\epsilon}\exp\left\{\frac{\lambda^2-2i\lambda t-t^2}{4\epsilon^2}-i\epsilon t+c^2\epsilon^2+\lambda\epsilon\right\}\left[\mathrm{erf}\left(c\epsilon+\frac{\lambda}{2\epsilon}-\frac{it}{2\epsilon}\right)\right] \\
&= e^{i\epsilon t-c^2\epsilon^2-\lambda\epsilon}\exp\left\{-\left(\frac{t}{2\epsilon}+i\left(c\epsilon+\frac{\lambda}{2\epsilon}\right)\right)^2\right\}\left[\mathrm{erf}\left(c\epsilon+\frac{\lambda}{2\epsilon}-\frac{it}{2\epsilon}\right)\right] \\
&= e^{i\epsilon t-c^2\epsilon^2-\lambda\epsilon}\left[\exp\left\{-\left(\frac{t}{2\epsilon}+i\left(c\epsilon+\frac{\lambda}{2\epsilon}\right)\right)^2\right\}-w\left(\frac{t}{2\epsilon}+i\left(c\epsilon+\frac{\lambda}{2\epsilon}\right)\right)\right] \\
&= \exp\left\{\frac{\lambda^2-2i\lambda t-t^2}{4\epsilon^2}\right\}-\exp\left\{i\epsilon t-c^2\epsilon^2-\lambda\epsilon\right\}w\left(\frac{t}{2\epsilon}+i\left(c\epsilon+\frac{\lambda}{2\epsilon}\right)\right).
\end{aligned} \tag{5.16}$$

Finally for F_4:

$$\begin{aligned}
F_4 &= \exp\left\{\frac{\lambda^2 - 2i\lambda t - t^2}{4\epsilon^2}\right\}\left[\operatorname{erf}\left(c\epsilon - \frac{\lambda}{2\epsilon} + \frac{it}{2\epsilon}\right)\right] \\
&= e^{-i\epsilon t - c^2\epsilon^2 + \lambda\epsilon} e^{i\epsilon t + c^2\epsilon^2 - \lambda\epsilon} \exp\left\{\frac{\lambda^2 - 2i\lambda t - t^2}{4\epsilon^2}\right\}\left[\operatorname{erf}\left(c\epsilon - \frac{\lambda}{2\epsilon} + \frac{it}{2\epsilon}\right)\right] \\
&= e^{-i\epsilon t - c^2\epsilon^2 + \lambda\epsilon} \exp\left\{\frac{\lambda^2 - 2i\lambda t - t^2}{4\epsilon^2} + i\epsilon t + c^2\epsilon^2 - \lambda\epsilon\right\}\left[\operatorname{erf}\left(c\epsilon - \frac{\lambda}{2\epsilon} + \frac{it}{2\epsilon}\right)\right] \\
&= e^{-i\epsilon t - c^2\epsilon^2 + \lambda\epsilon} \exp\left\{-\left(-\frac{t}{2\epsilon} + i\left(c\epsilon - \frac{\lambda}{2\epsilon}\right)\right)^2\right\}\left[\operatorname{erf}\left(c\epsilon - \frac{\lambda}{2\epsilon} + \frac{it}{2\epsilon}\right)\right] \\
&= e^{-i\epsilon t - c^2\epsilon^2 + \lambda\epsilon}\left[\exp\left\{\frac{\lambda^2 - 2i\lambda t - t^2}{4\epsilon^2} + i\epsilon t + c^2\epsilon^2 - \lambda\epsilon\right\} - w\left(-\frac{t}{2\epsilon} + i\left(c\epsilon - \frac{\lambda}{2\epsilon}\right)\right)\right] \\
&= \exp\left\{\frac{\lambda^2 - 2i\lambda t - t^2}{4\epsilon^2}\right\} - \exp\left\{-i\epsilon t - c^2\epsilon^2 + \lambda\epsilon\right\} w\left(-\frac{t}{2\epsilon} + i\left(c\epsilon - \frac{\lambda}{2\epsilon}\right)\right). \qquad (5.17)
\end{aligned}$$

Combining the above, we can express equation (5.12) as:

$$\begin{aligned}
I_\epsilon &= \frac{\sqrt{\pi}}{4\epsilon}\left[2\exp\left\{\frac{\lambda^2 + 2i\lambda t - t^2}{4\epsilon^2}\right\} + 2\exp\left\{\frac{\lambda^2 - 2i\lambda t - t^2}{4\epsilon^2}\right\}\right. \\
&- \exp\left\{i\epsilon t - c^2\epsilon^2 + \lambda\epsilon\right\} w\left(\frac{t}{2\epsilon} + i\left(c\epsilon - \frac{\lambda}{2\epsilon}\right)\right) \\
&- \exp\left\{-i\epsilon t - c^2\epsilon^2 - \lambda\epsilon\right\} w\left(-\frac{t}{2\epsilon} + i\left(c\epsilon + \frac{\lambda}{2\epsilon}\right)\right) \\
&- \exp\left\{i\epsilon t - c^2\epsilon^2 - \lambda\epsilon\right\} w\left(\frac{t}{2\epsilon} + i\left(c\epsilon + \frac{\lambda}{2\epsilon}\right)\right) \\
&- \left.\exp\left\{-i\epsilon t - c^2\epsilon + \lambda\epsilon\right\} w\left(-\frac{t}{2\epsilon} + i\left(c\epsilon - \frac{\lambda}{2\epsilon}\right)\right)\right]. \qquad (5.18)
\end{aligned}$$

Bringing terms together, and noting that $w(-x+iy) = \overline{w(x+iy)}$ we can write:

$$\begin{aligned}
I_\epsilon &= \frac{\sqrt{\pi}}{4\epsilon}\left[4\exp\left\{\frac{\lambda^2-t^2}{4\epsilon^2}\right\}\left(\cos\left(\frac{\lambda t}{2\epsilon^2}\right)\right)\right.\\
&- \exp\left\{-c^2\epsilon^2+\lambda\epsilon\right\}\left\{e^{i\epsilon t}w\left(\frac{t}{2\epsilon}+i\left(c\epsilon-\frac{\lambda}{2\epsilon}\right)\right)+e^{-i\epsilon t}\overline{w\left(\frac{t}{2\epsilon}+i\left(c\epsilon-\frac{\lambda}{2\epsilon}\right)\right)}\right\}\\
&- \left.\exp\left\{-c^2\epsilon^2-\lambda\epsilon\right\}\left\{e^{-i\epsilon t}\overline{w\left(\frac{t}{2\epsilon}+i\left(c\epsilon+\frac{\lambda}{2\epsilon}\right)\right)}+e^{i\epsilon t}w\left(\frac{t}{2\epsilon}+i\left(c\epsilon+\frac{\lambda}{2\epsilon}\right)\right)\right\}\right].
\end{aligned} \tag{5.19}$$

Let $z^- = \frac{t}{2\epsilon}+i\left(c\epsilon-\frac{\lambda}{2\epsilon}\right)$ and $z^+ = \frac{t}{2\epsilon}+i\left(c\epsilon+\frac{\lambda}{2\epsilon}\right)$, then:

$$\begin{aligned}
I_\epsilon &= \frac{\sqrt{\pi}}{4\epsilon}\left[4\exp\left\{\frac{\lambda^2-t^2}{4\epsilon^2}\right\}\left(\cos\left(\frac{\lambda t}{2\epsilon^2}\right)\right)\right.\\
&- \exp\left\{-c^2\epsilon^2+\lambda\epsilon\right\}\left\{e^{i\epsilon t}w\left(z^-\right)+e^{-i\epsilon t}\overline{w\left(z^-\right)}\right\}\\
&- \left.\exp\left\{-c^2\epsilon^2-\lambda\epsilon\right\}\left\{e^{-i\epsilon t}\overline{w\left(z^+\right)}+e^{i\epsilon t}w\left(z^+\right)\right\}\right]\\
&= \frac{\sqrt{\pi}}{\epsilon}\exp\left\{\frac{\lambda^2-t^2}{4\epsilon^2}\right\}\cos\left(\frac{\lambda t}{2\epsilon^2}\right)\\
&- \frac{\sqrt{\pi}}{4\epsilon}\exp\left\{-c^2\epsilon^2+\lambda\epsilon\right\}\left\{2\cos\left(\epsilon t\right)\Re\left[w\left(z^-\right)\right]-2\sin\left(\epsilon t\right)\Im\left[w\left(z^-\right)\right]\right\}\\
&- \frac{\sqrt{\pi}}{4\epsilon}\exp\left\{-c^2\epsilon^2-\lambda\epsilon\right\}\left\{2\cos\left(\epsilon t\right)\Re\left[w\left(z^+\right)\right]-2\sin\left(\epsilon t\right)\Im\left[w\left(z^+\right)\right]\right\}.
\end{aligned} \tag{5.20}$$

Combining the above, we can express equation (5.5) as:

$$\begin{aligned}
C_1 + C_3 &= \frac{2\sqrt{\pi}}{\epsilon} e^{\lambda^2/4\epsilon^2} e^{-t^2/4\epsilon^2} \cos\left(\frac{\lambda t}{2\epsilon^2}\right) - \frac{\sqrt{\pi}}{\epsilon} \exp\left\{\frac{\lambda^2 - t^2}{4\epsilon^2}\right\} \cos\left(\frac{\lambda t}{2\epsilon^2}\right) \\
&+ \frac{\sqrt{\pi}}{2\epsilon} \exp\left\{-c^2\epsilon^2 + \lambda\epsilon\right\} \left\{\cos(\epsilon t) \Re\left[w\left(z^-\right)\right] - \sin(\epsilon t) \Im\left[w\left(z^-\right)\right]\right\} \\
&+ \frac{\sqrt{\pi}}{2\epsilon} \exp\left\{-c^2\epsilon^2 - \lambda\epsilon\right\} \left\{\cos(\epsilon t) \Re\left[w\left(z^+\right)\right] - \sin(\epsilon t) \Im\left[w\left(z^+\right)\right]\right\} \\
&= \frac{\sqrt{\pi}}{\epsilon} \exp\left\{\frac{\lambda^2 - t^2}{4\epsilon^2}\right\} \cos\left(\frac{\lambda t}{2\epsilon^2}\right) \\
&+ \frac{\sqrt{\pi}}{2\epsilon} \exp\left\{-c^2\epsilon^2 + \lambda\epsilon\right\} \left\{\cos(\epsilon t) \Re\left[w\left(z^-\right)\right] - \sin(\epsilon t) \Im\left[w\left(z^-\right)\right]\right\} \\
&+ \frac{\sqrt{\pi}}{2\epsilon} \exp\left\{-c^2\epsilon^2 - \lambda\epsilon\right\} \left\{\cos(\epsilon t) \Re\left[w\left(z^+\right)\right] - \sin(\epsilon t) \Im\left[w\left(z^+\right)\right]\right\}
\end{aligned} \tag{5.21}$$

Finally, we calculate C_2.

$$\begin{aligned}
C_2 &= \int_{-c}^{c} e^{-c\epsilon^2|x|} \left(\frac{e^{\lambda x} + e^{-\lambda x}}{2}\right) e^{ixt}\,\mathrm{d}x \\
&= \int_{0}^{c} e^{-c\epsilon^2 x} \left(\frac{e^{\lambda x} + e^{-\lambda x}}{2}\right) e^{ixt}\,\mathrm{d}x + \int_{-c}^{0} e^{c\epsilon^2 x} \left(\frac{e^{\lambda x} + e^{-\lambda x}}{2}\right) e^{ixt}\,\mathrm{d}x \\
&= \int_{0}^{c} \frac{e^{\left(it+\lambda-c\epsilon^2\right)x} + e^{\left(it-\lambda-c\epsilon^2\right)x}}{2}\,\mathrm{d}x + \int_{-c}^{0} \frac{e^{\left(it+\lambda+c\epsilon^2\right)x} + e^{\left(it-\lambda+c\epsilon^2\right)x}}{2}\,\mathrm{d}x \\
&= \frac{1}{2}\left[\frac{e^{\left(it+\lambda-c\epsilon^2\right)x}}{it+\lambda-c\epsilon^2} + \frac{e^{\left(it-\lambda-c\epsilon^2\right)x}}{it-\lambda-c\epsilon^2}\right]_0^c + \frac{1}{2}\left[\frac{e^{\left(it+\lambda+c\epsilon^2\right)x}}{it+\lambda+c\epsilon^2} + \frac{e^{\left(it-\lambda+c\epsilon^2\right)x}}{it-\lambda+c\epsilon^2}\right]_{-c}^0
\end{aligned} \tag{5.22}$$

Writing $\alpha = \lambda + c\epsilon^2$ and $\beta = \lambda - c\epsilon^2$, we can express equation (5.22) as:

$$
\begin{aligned}
C_2 &= \frac{1}{2}\left[\frac{e^{(it+\beta)c}-1}{it+\beta}+\frac{e^{(it-\alpha)c}-1}{it-\alpha}+\frac{1-e^{-(it+\alpha)c}}{it+\alpha}+\frac{1-e^{-(it-\beta)c}}{it-\beta}\right] \\
&= \frac{(\alpha-\beta)\,t^2+\alpha\beta\,(\beta-\alpha)+\left[t^3\left(e^{-c\alpha}-e^{c\beta}\right)+(\alpha-\beta)\,t\left(\alpha e^{c\beta}-\beta e^{-c\alpha}\right)\right]\sin(ct)}{t^4+(\alpha^2+\beta^2)\,t^2+\alpha^2\beta^2} \\
&+ \frac{\left[t^2\left(\beta e^{c\beta}-\alpha e^{-c\alpha}\right)+\alpha\beta\left(\alpha e^{c\beta}-\beta e^{-c\alpha}\right)\right]\cos(ct)}{t^4+(\alpha^2+\beta^2)\,t^2+\alpha^2\beta^2}.
\end{aligned} \tag{5.23}
$$

Combining the above, we have:

$$
\begin{aligned}
h_2(t) &= C_1+C_2+C_3 \\
&= \frac{\sqrt{\pi}}{\epsilon}\exp\left\{\frac{\lambda^2-t^2}{4\epsilon^2}\right\}\cos\left(\frac{\lambda t}{2\epsilon^2}\right) \\
&+ \frac{\sqrt{\pi}}{2\epsilon}\exp\left\{-c^2\epsilon^2+\lambda\epsilon\right\}\left\{\cos(\epsilon t)\,\Re\left[w\left(z^-\right)\right]-\sin(\epsilon t)\,\Im\left[w\left(z^-\right)\right]\right\} \\
&+ \frac{\sqrt{\pi}}{2\epsilon}\exp\left\{-c^2\epsilon^2-\lambda\epsilon\right\}\left\{\cos(\epsilon t)\,\Re\left[w\left(z^+\right)\right]-\sin(\epsilon t)\,\Im\left[w\left(z^+\right)\right]\right\} \\
&+ \frac{(\alpha-\beta)\,t^2+\alpha\beta\,(\beta-\alpha)+\left[t^3\left(e^{-c\alpha}-e^{c\beta}\right)+(\alpha-\beta)\,t\left(\alpha e^{c\beta}-\beta e^{-c\alpha}\right)\right]\sin(ct)}{t^4+(\alpha^2+\beta^2)\,t^2+\alpha^2\beta^2} \\
&+ \frac{\left[t^2\left(\beta e^{c\beta}-\alpha e^{-c\alpha}\right)+\alpha\beta\left(\alpha e^{c\beta}-\beta e^{-c\alpha}\right)\right]\cos(ct)}{t^4+(\alpha^2+\beta^2)\,t^2+\alpha^2\beta^2},
\end{aligned} \tag{5.24}
$$

which has the following graph:

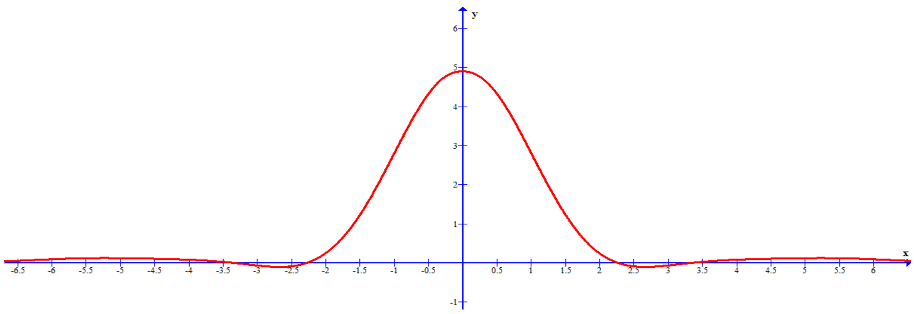

Figure 5.2: Graph of h_2 with $c=2$, $\epsilon=1$ and $\lambda=\pi/2$.

We note that the Faddeeva function, as introduced in equation (5.13), contain a Gaussian of a complex variable. For our example the variables z^- and z^+ are variables in t only, since the imaginary parts are constants. Hence, the Faddeeva terms in equation (5.24) have Gaussian decay in t. When we combine this with the very small coefficients ($\exp\left\{-c^2\epsilon^2+\lambda\epsilon\right\}=0.088$ and $\exp\left\{-c^2\epsilon^2-\lambda\epsilon\right\}=0.004$), we deduce that these terms are insignificant in comparison to the other terms.

It is clear that the function $h_2(t)$ becomes insignificant outside some closed interval, i.e. it has compact essential numerical support.

The third function we will consider is the function $B_3(x)=\operatorname{sech}(\alpha x)$ that we introduced in Chapter 4. We demonstrated that the inverse Fourier transform b_3 of B_3 belongs to the space $F_{[\lambda,p]}$. We will now calculate the relaxation spectrum h_3:

$$\begin{aligned}
h_3(y)=\mathscr{F}^{-1}\left[B_3(x)\cdot\cosh(\lambda x)\right] &= \int_{-\infty}^{\infty}\operatorname{sech}(\alpha x)\cosh(\lambda x)\,e^{ixy}\,\mathrm{d}x\\
&= -\frac{\pi\alpha Be^{y\pi}}{\left[1-Ae^{y\pi}\right]}\left[\frac{\sin(\pi\lambda/2)\sinh(y\pi/2)}{\cos(\lambda\pi)+\cosh(y\pi)}\right]\\
&+ \frac{2\pi}{\alpha\left[1-Ae^{y\pi}\right]}e^{\pi y/2\alpha}\cos\left(\frac{\pi\lambda}{2\alpha}\right), \qquad (5.25)
\end{aligned}$$

where $A=\frac{\cos(\lambda\pi)}{\cos(\alpha\pi)}$ and $B=\frac{\sin(\lambda\pi)}{\cos(\alpha\pi)}$(see Fourier transform 3 in appendix for calculations).

$h_3=h_3(y)$ has the following graph:

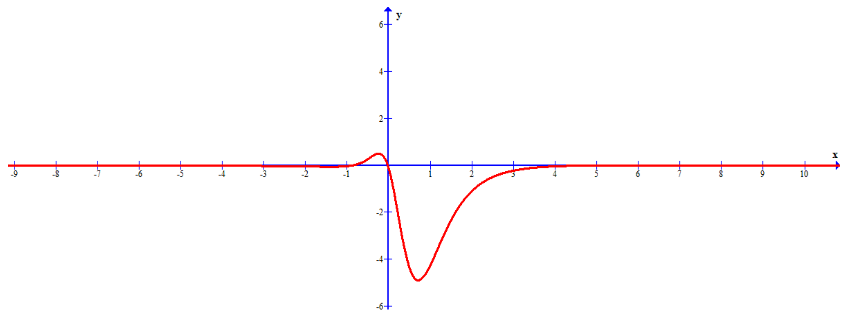

Figure 5.3: Graph of h_3 with $\alpha = 3$ and $\lambda = \pi/2$.

Once again we can see that the relaxation spectrum $h_3(y)$ becomes insignificant outside some closed interval.

Finally, we calculate the relaxation spectrum for the fourth function that we introduced in the previous chapter, $B_4(x)$. We demonstrated that the inverse Fourier transform b_4 of B_4 belonged to the space $F_{[\lambda,p]}$. Now we calculate h_4.

$$
\begin{aligned}
h_4(y) &= \mathscr{F}^{-1}\left[B_4(x)\cdot\cosh(\lambda x)\right] \\
&= \int_{-\beta}^{\beta}\left(\beta^2 - x^2\right)\cosh(\lambda x)\, e^{ixy}\,\mathrm{d}x \\
&= \beta^2\int_{-\beta}^{\beta}\frac{e^{\lambda x}+e^{-\lambda x}}{2}e^{ixy}\,\mathrm{d}x - \int_{-\beta}^{\beta}x^2\frac{e^{\lambda x}+e^{-\lambda x}}{2}e^{ixy}\,\mathrm{d}x \\
&= \frac{\beta^2}{2}\int_{-\beta}^{\beta}e^{x(\lambda+iy)}+e^{x(iy-\lambda)}\,\mathrm{d}x - \frac{1}{2}\int_{-\beta}^{\beta}x^2e^{x(\lambda+iy)}+e^{x(iy-\lambda)}\,\mathrm{d}x \\
&= \frac{\beta^2}{2}\left[\frac{e^{x(\lambda+iy)}}{\lambda+iy}+\frac{e^{x(iy-\lambda)}}{iy-\lambda}\right]_{-\beta}^{\beta} - \frac{1}{2}\int_{-\beta}^{\beta}x^2e^{x(\lambda+iy)}+e^{x(iy-\lambda)}\,\mathrm{d}x \\
&= \frac{\beta^2}{2}\left[\frac{e^{\beta(\lambda+iy)}}{\lambda+iy}+\frac{e^{\beta(iy-\lambda)}}{iy-\lambda}-\frac{e^{-\beta(\lambda+iy)}}{\lambda+iy}-\frac{e^{-\beta(iy-\lambda)}}{iy-\lambda}\right]-\frac{1}{2}E_1, \qquad (5.26)
\end{aligned}
$$

where:

$$E_1 = \int_{-\beta}^{\beta} x^2 e^{\rho x} + e^{\sigma x}\, \mathrm{d}x,$$

with $\rho = \lambda + iy$ and $\sigma = iy - \lambda$.

Using results from equations (4.40) and (4.41), we can write

$$\begin{aligned}\int_{-\beta}^{\beta} x^2 e^{ax}\, \mathrm{d}x &= \left(\frac{\beta^2}{a} + \frac{2}{a^3}\right)\left(e^{\beta a} - e^{-\beta a}\right) - \frac{2\beta}{a^2}\left(e^{\beta a} + e^{-\beta a}\right) \\ &= \left(\frac{2\beta^2}{a} + \frac{4}{a^3}\right)\sinh(\beta a) - \frac{4\beta}{a^2}\cosh(\beta a)\,. \end{aligned} \tag{5.27}$$

If follows that

$$\begin{aligned} E_1 &= \left(\frac{2\beta^2}{\rho} + \frac{4}{\rho^3}\right)\sinh(\beta\rho) - \frac{4\beta}{\rho^2}\cosh(\beta\rho) \\ &+ \left(\frac{2\beta^2}{\sigma} + \frac{4}{\sigma^3}\right)\sinh(\beta\sigma) - \frac{4\beta}{\sigma^2}\cosh(\beta\sigma)\,. \end{aligned} \tag{5.28}$$

We substitute $\rho = \lambda + iy$ and $\sigma = iy - \lambda$ back into equation (5.25):

$$\begin{aligned} E_1 &= \left(\frac{2\beta^2}{\lambda + iy} + \frac{4}{(\lambda + iy)^3}\right)\sinh(\beta(\lambda + iy)) - \frac{4\beta}{(\lambda + iy)^2}\cosh(\beta(\lambda + iy)) \\ &+ \left(\frac{2\beta^2}{iy - \lambda} + \frac{4}{(iy - \lambda)^3}\right)\sinh(\beta(iy - \lambda)) - \frac{4\beta}{(iy - \lambda)^2}\cosh(\beta(iy - \lambda)) \end{aligned} \tag{5.29}$$

Combining the above we have that h_4 is:

$$\begin{aligned}
h_4(y) &= \frac{\beta^2}{2}\left[\frac{e^{\beta(\lambda+iy)}-e^{-\beta(\lambda+iy)}}{\lambda+iy}+\frac{e^{\beta(iy-\lambda)}-e^{-\beta(iy-\lambda)}}{iy-\lambda}\right]-\frac{1}{2}E_1 \\
&= \beta^2\left[\frac{\sinh(\beta(\lambda+iy))}{\lambda+iy}+\frac{\sinh(\beta(iy-\lambda))}{iy-\lambda}\right] \\
&- \left[\left(\frac{\beta^2}{\lambda+iy}+\frac{2}{(\lambda+iy)^3}\right)\sinh(\beta(\lambda+iy))-\frac{2\beta}{(\lambda+iy)^2}\cosh(\beta(\lambda+iy))\right. \\
&+ \left.\left(\frac{\beta^2}{iy-\lambda}+\frac{2}{(iy-\lambda)^3}\right)\sinh(\beta(iy-\lambda))-\frac{2\beta}{(iy-\lambda)^2}\cosh(\beta(iy-\lambda))\right] \\
&= \frac{2\beta}{(\lambda+iy)^2}\cosh(\beta(\lambda+iy))-\frac{2}{(\lambda+iy)^3}\sinh(\beta(\lambda+iy)) \\
&+ \frac{2\beta}{(iy-\lambda)^2}\cosh(\beta(iy-\lambda))-\frac{2}{(iy-\lambda)^3}\sinh(\beta(iy-\lambda)). \qquad (5.30)
\end{aligned}$$

We use double angle formulae to simplify equation (5.30);

$$\begin{aligned}
h_4(y) &= \frac{2\beta}{(\lambda+iy)^2}\left(\cosh(\beta\lambda)\cos(\beta y)+i\sinh(\beta\lambda)\sin(\beta y)\right) \\
&- \frac{2}{(\lambda+iy)^3}\left(\sinh(\beta\lambda)\cos(\beta y)+i\cosh(\beta\lambda)\sin(\beta y)\right) \\
&+ \frac{2\beta}{(iy-\lambda)^2}\left(\cosh(\beta\lambda)\cos(\beta y)-i\sinh(\beta\lambda)\sin(\beta y)\right) \\
&- \frac{2}{(iy-\lambda)^3}\left(i\cosh(\beta\lambda)\sin(\beta y)-\sinh(\beta\lambda)\cos(\beta y)\right). \qquad (5.31)
\end{aligned}$$

Collecting similar terms:

$$\begin{aligned} h_4(y) &= \left(\frac{2\beta}{(\lambda+iy)^2}+\frac{2\beta}{(iy-\lambda)^2}\right)\cosh(\beta\lambda)\cos(\beta y) \\ &+ \left(\frac{2}{(iy-\lambda)^3}-\frac{2}{(\lambda+iy)^3}\right)\sinh(\beta\lambda)\cos(\beta y) \\ &+ \left(\frac{2\beta}{(\lambda+iy)^2}-\frac{2\beta}{(iy-\lambda)^2}\right)i\sinh(\beta\lambda)\sin(\beta y) \\ &- \left(\frac{2}{(\lambda+iy)^3}+\frac{2}{(iy-\lambda)^3}\right)i\cosh(\beta\lambda)\sin(\beta y). \end{aligned} \tag{5.32}$$

Simplifying:

$$\begin{aligned} h_4(y) &= \left(\frac{4\beta(\lambda^2-y^2)}{(\lambda+iy)^2(iy-\lambda)^2}\right)\cosh(\beta\lambda)\cos(\beta y) \\ &+ \left(\frac{4\lambda(\lambda^2-3y^2)}{(iy-\lambda)^3(\lambda+iy)^3}\right)\sinh(\beta\lambda)\cos(\beta y) \\ &- \left(\frac{8i\beta\lambda y}{(\lambda+iy)^2(iy-\lambda)^2}\right)i\sinh(\beta\lambda)\sin(\beta y) \\ &- \left(\frac{4iy(3\lambda^2-y^2)}{(iy-\lambda)^3(\lambda+iy)^3}\right)i\cosh(\beta\lambda)\sin(\beta y). \end{aligned} \tag{5.33}$$

Which we write as

$$\begin{aligned} h_4(y) &= 4\beta\left(\frac{(\lambda^2-y^2)\cosh(\beta\lambda)\cos(\beta y)+2\lambda y\sinh(\beta\lambda)\sin(\beta y)}{(\lambda+iy)^2(iy-\lambda)^2}\right) \\ &+ 4\left(\frac{\lambda(\lambda^2-3y^2)\sinh(\beta\lambda)\cos(\beta y)+y(3\lambda^2-y^2)\cosh(\beta\lambda)\sin(\beta y)}{(iy-\lambda)^3(\lambda+iy)^3}\right) \\ &= 4\beta\left(\frac{(\lambda^2-y^2)C\cos(\beta y)+2\lambda yS\sin(\beta y)}{(\lambda^2+y^2)^2}\right) \\ &- 4\left(\frac{\lambda(\lambda^2-3y^2)S\cos(\beta y)+y(3\lambda^2-y^2)C\sin(\beta y)}{(y^2+\lambda^2)^3}\right), \end{aligned} \tag{5.34}$$

where $C = \cosh(\beta\lambda)$ and $S = \sinh(\beta\lambda)$. The function h_4 has the following graph:

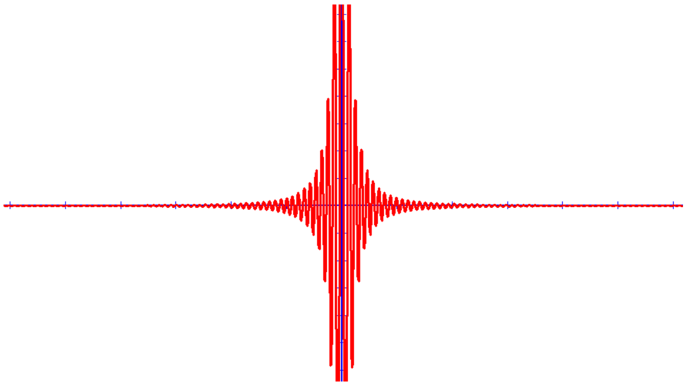

Figure 5.4: Graph of h_4 with $\beta = 3$ and $\lambda = 2$.

From the four possible relaxation spectra that we have calculated above we can see that none of them have compact support. However, what we do observe is that outside some closed interval each relaxation spectrum becomes (visually) insignificant. We will perform calculations in the following sections that agree with our observations above. As a remark we note that the rate of decay at infinity differs for the four functions we have calculated above. For h_1 we note that the decay at infinity is Gaussian decay and we remind the reader that the function B_1 which represents $\hat{g}$ in our problem, has Gaussian decay at infinity. For h_2, we have terms with Gaussian decay but also terms with polynomial decay, while B_2 had Gaussian decay at infinity but exponential decay locally. For h_3, we see that the decay at infinity is of the order $e^{a|y|}$ and we remind the reader that the function B_3 that we introduced in Chapter 4 was a sech function, which has decay at infinity of the order $e^{a|y|}$. Finally, for h_4, we have polynomial decay at infinity. If we refer back to the function we defined as B_4 in Chapter 4, we see this function had compact support and so was zero at the far field, however, we do note that the function was a polynomial. We note the discontinuity of B_4', which might play a role. It would seem that there is a link between the rate of decay of the relaxation spectrum h compared with $\hat{g}$.

So we ask the question; can we find an interval of "compact essential numerical support" where the behaviour of the relaxation spectrum outside this interval is insignificant?

In the work of Grip and Pfander [35], we see a similar concept, where they refer to "essentially compact" support as some "reasonably small" compact set in which the function decays fast enough to ensure that in any practical application, the function values outside this set are very small compared to the overall noise level and therefore negligible.

One approach could be to find an interval for which the integral over this interval of the relaxation spectrum gives us 99% of some norm of the relaxation spectrum. This immediately introduces further questions; that is, what is the best norm to use and what spaces of functions should we be working in? In the next section we consider a variety of possible norms and demonstrate their suitability by applying them to the Gaussian function.

5.3 Compact Essential Numerical Support: Definitions and Examples

We begin this section by making a formal definition of what we mean by compact essential numerical support.

Definition 5.1. *Compact Essential Numerical Support*

For a given β (small), a function f has compact essential numerical support on an interval $A_\beta = [-a(\beta), a(\beta)]$ if:

$$\left\| f|_{\mathbb{R}\setminus A_\beta} \right\|_{\sup} \leq \beta. \tag{5.35}$$

We note that in the remainder of this chapter, when referring to an interval of compact essential numerical support as $[-a, a]$, we mean $[-a(\beta), a(\beta)]$, where the interval of support depends on the tolerance β.

We could extend the idea of compact essential numerical support to other norms i.e. we could consider a weighted integral norm, defined as:

$$\|f\|_w = \int_{\mathbb{R}} w(x) |f(x)| \, \mathrm{d}x, \tag{5.36}$$

where w is a positive weight function. Then we could calculate $C = [-c, c]$ such that

$$\left\| f|_{\mathbb{R}\setminus C} \right\|_w \leq \beta. \tag{5.37}$$

Another possible definition for compact essential numerical support is as follows: we may wish to calculate an interval $C = [-c, c]$ such that:

$$\| f|_C \| \leq \alpha \, \|f\| \tag{5.38}$$

for $0 < \alpha < 1$, for α very close to 1.

We need not consider symmetric intervals only as intervals of compact essential numerical support. However, we have done so for simplicity, in the calculations that follow. In the calculations that follow, we demonstrate different ways we might like to calculate an interval of compact essential numerical support using the Gaussian function as an example.

We have seen in the calculations above that the Gaussian function is a possible candidate for the relaxation spectrum. The Gaussian functions do not have compact support, however they do decay rapidly outside a closed interval.

That is, $h(t)$ is of the form $e^{-\alpha^2 t^2}$.

So we could ask the question:

What value does a need to be to ensure that:

$$\int_{-a}^{a} |h(t)| \, \mathrm{d}t = 0.99 \int_{-\infty}^{\infty} |h(t)| \, \mathrm{d}t, \tag{5.39}$$

where here we are making use of the L^1 norm.

Since h is a Gaussian, it follows that h is always positive, hence we can write equation (5.39) as:

$$\int_{-a}^{a} h(t)\ \mathrm{d}t = 0.99 \int_{-\infty}^{\infty} h(t)\ \mathrm{d}t \tag{5.40}$$

and if we let $h(t) = e^{-\alpha^2 t^2}$, then we can write equation (5.27) as:

$$\int_{-a}^{a} e^{-\alpha^2 t^2}\ \mathrm{d}t = 0.99 \int_{-\infty}^{\infty} e^{-\alpha^2 t^2}\ \mathrm{d}t. \tag{5.41}$$

We can determine a from standard tables for the Normal distribution. In fact if $h(t)$ represents the pdf of a normal distribution, with random variable T, then we can deduce that it would have a normal distribution with mean zero and variance $1/2\alpha^2$. We can deduce from Normal tables that;

$$P\left(-2.57583 \leq \frac{T-0}{1/\alpha\sqrt{2}} \leq 2.57583\right) \quad = \quad 0.99,$$

that is:

$$P\left(-\frac{1.82138686}{\alpha} \leq T \leq \frac{1.82138686}{\alpha}\right) \quad = \quad 0.99. \tag{5.42}$$

It follows that "99% of the support", in terms of the L^1 norm, of $h(t) = e^{-\alpha^2 t^2}$ is contained in the interval $[-1.8214/\alpha, 1.8214/\alpha]$.

Since we have defined h as belonging to $L^q(\mathbb{R})$, we should also consider the L^q-norm of h.

We ask the question; what value does b need to be to ensure that:

$$\int_{-b}^{b} |h(t)|^q \, \mathrm{d}t = 0.99 \, \|h\|_q^q? \tag{5.43}$$

We keep h as defined above: $h(t) = e^{-\alpha^2 t^2}$, then equation (5.30) becomes:

$$\int_{-b}^{b} \left| e^{-\alpha^2 t^2} \right|^q \, \mathrm{d}t = \int_{-b}^{b} e^{-q\alpha^2 t^2} \, \mathrm{d}t. \tag{5.44}$$

Once again, we have a pdf of a normal distribution, and so we can deduce the value of b by using Normal tables for confidence intervals. We define T as the random variable relating to the pdf in equation (5.44), then the standard deviation of this distribution is $1/\alpha\sqrt{2q}$ and from Normal tables we obtain:

$$P\left(-2.57583 \leq \frac{T-0}{1/\alpha\sqrt{2q}} \leq 2.57583\right) = 0.99.$$

That is:

$$P\left(-\frac{1.82138686}{\alpha\sqrt{q}} \leq T \leq \frac{1.82138686}{\alpha\sqrt{q}}\right) \quad = \quad 0.99. \tag{5.45}$$

It follows that “99% of the support", in terms of the L^q norm (to the power q), of $h\left(t\right) = e^{-\alpha^2 t^2}$ is contained in the interval $\left[-1.8214/\alpha\sqrt{q}, 1.8214/\alpha\sqrt{q}\right]$.

Finally, we will make use of the approach we introduced in Definition 5.1, that is, we will calculate an interval $[-c, c]$ such that $\left\| h|_{\mathbb{R}\setminus[-c,c]} \right\|_{\sup} \leq \beta$. Suppose we want β to take the value 0.1. Gaussian functions are symmetric and decrease monotonically either side of their maximum value. Hence, for $h\left(t\right) = e^{-\alpha^2 t^2}$, to calculate c, we solve $e^{-\alpha^2 c^2} = 0.1$. This corresponds to:

$$c = \pm\frac{\sqrt{-\ln\left(0.1\right)}}{\alpha} = \pm\frac{1.52}{\alpha}. \tag{5.46}$$

That is, h has compact essential numerical support in the interval $[-1.52/\alpha, 1.52/\alpha]$. If we compare this with the interval we obtained for the L^1 norm: $[-1.8214/\alpha, 1.8214/\alpha]$, we can see that there is not a great difference between the two (for this example).

Finally, in this section we make some remarks with regards to the weighted norm that we introduced at the beginning of this section (see equation (5.36)):

$$\|f\|_w = \int_{\mathbb{R}} w\left(x\right)\left|f\left(x\right)\right|\,\mathrm{d}x \tag{5.47}$$

for $w > 0$. Now, in order to probe a little further into our relaxation spectra, we might wish to consider an exponentially growing weight function, e.g. we might set $w(x) = e^{a|x|}$. The reason this may allow us to deduce further properties of our relaxation spectra is that, if we refer back to the examples at the beginning of this chapter, we saw that our relaxation spectra had different types of decay at far field. Some had Gaussian decay while others had decay of the form $e^{-b|x|}$ and also polynomial decay. Hence, in evaluating this norm to estimate an interval of compact essential numerical support we might want to consider two cases, where:

1. Case 1: $w(x) = e^{a|x|}$ for some fixed $a \in \mathbb{R}$, and
2. Case 2: $w(x) = e^{a|x|}$ for $0 < a < a_{max}$.

Clearly, there are functions in $F_{[\lambda,p]}$ such that the weighted norm of the corresponding relaxation spectrum: $\|h\|_w = \infty$. However, this norm may be useful in gaining a deeper insight into the types of functions we are dealing with.

In this section we have performed various calculations, with different norms to estimate the compact essential numerical support of a Gaussian function. However there are many other types of functions that could be in this space. We wish to extend our analysis to "typical" elements of $F_{[\lambda,p]}$.

In the next section we introduce a possible approach for deducing an interval of compact essential numerical support for the relaxation spectrum.

5.4 Estimating the Compact Essential Numerical Support

We would like to find a result that agrees with what our graphs have shown involving estimates using the supremum norm.

In this section we present a method, using the supremum norm, to estimate the compact essential numerical support of a general function in $F_{[\lambda,p]}$. Our approach is to split the relaxation spectrum into two separate functions. One function represents the inverse Fourier transform of $\hat{g}\xi_\lambda$ at the far field where we assume the function has Gaussian decay. The other part represents the inverse Fourier transform of $\hat{g}\xi_\lambda$ on a closed interval where we assume that this function contributes the most to the relaxation spectrum. We consider each function in turn, starting with the far field contribution, which involves extensive calculations, where we make use of Faddeeva functions.

As we have seen earlier on in this thesis, the function $\hat{g}$ must have exponential decay, or be zero outside some closed interval to ensure that $\hat{g}\cosh$ belongs to $L^p(\mathbb{R})$. We could therefore represent the relaxation spectrum h in the following form:

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} \hat{g}(x)\cosh(\lambda x)\, e^{itx}\, \mathrm{d}x \\ &= \int_{-\infty}^{-c} \hat{g}(x)\cosh(\lambda x)\, e^{itx}\, \mathrm{d}x + \int_{-c}^{c} \hat{g}(x)\cosh(\lambda x)\, e^{itx}\, \mathrm{d}x \\ &+ \int_{c}^{\infty} \hat{g}(x)\cosh(\lambda x)\, e^{itx}\, \mathrm{d}x, \end{aligned} \tag{5.48}$$

where $\hat{g}$ is a Gaussian or is zero outside $[-c, c]$. We can express the relaxation spectrum h as:

$$h(t) = f(t) + j(t), \tag{5.49}$$

where

$$f(t) = \int_{|x|\geq c} e^{-\epsilon^2 x^2} \cosh(\lambda x) e^{itx} \, \mathrm{d}x, \tag{5.50}$$
$$j(t) = \int_{|x|<c} \hat{g}(x) \cosh(\lambda x) e^{itx} \, \mathrm{d}x. \tag{5.51}$$

We evaluate the integral in equation (5.50):

$$f(t) = \int_{-\infty}^{-c} e^{-\epsilon^2 x^2} \cosh(\lambda x) e^{itx} \, \mathrm{d}x + \int_{c}^{\infty} e^{-\epsilon^2 x^2} \cosh(\lambda x) e^{itx} \, \mathrm{d}x. \tag{5.52}$$

We note that, after some rearrangement, we can write:

$$e^{-\epsilon^2 x^2} \cosh(\lambda x) e^{itx} = \frac{1}{2} \exp\left\{\frac{a^2}{4\epsilon^2}\right\} \exp\left\{-\epsilon^2 r^2\right\} + \frac{1}{2} \exp\left\{\frac{b^2}{4\epsilon^2}\right\} \exp\left\{-\epsilon^2 s^2\right\}, \tag{5.53}$$

where $a = \lambda + it$, $b = it - \lambda$, $r = x - a/2\epsilon^2$ and $s = x - b/2\epsilon^2$.

Substituting this into equation (5.52)

$$
\begin{aligned}
f(t) &= \int_{-\infty-a/2\epsilon^2}^{-c-a/2\epsilon^2} \frac{1}{2}\exp\left\{\frac{a^2}{4\epsilon^2}\right\}\exp\left\{-\epsilon^2 r^2\right\}\,\mathrm{d}r \\
&+ \int_{-\infty-b/2\epsilon^2}^{-c-b/2\epsilon^2} \frac{1}{2}\exp\left\{\frac{b^2}{4\epsilon^2}\right\}\exp\left\{-\epsilon^2 s^2\right\}\,\mathrm{d}s \\
&+ \int_{c-a/2\epsilon^2}^{\infty-a/2\epsilon^2} \frac{1}{2}\exp\left\{\frac{a^2}{4\epsilon^2}\right\}\exp\left\{-\epsilon^2 r^2\right\}\,\mathrm{d}r \\
&+ \int_{c-b/2\epsilon^2}^{\infty-b/2\epsilon^2} \frac{1}{2}\exp\left\{\frac{b^2}{4\epsilon^2}\right\}\exp\left\{-\epsilon^2 s^2\right\}\,\mathrm{d}s, \qquad (5.54)
\end{aligned}
$$

which we express as:

$$
\begin{aligned}
f(t) &= \frac{1}{2}\exp\left\{\frac{a^2}{4\epsilon^2}\right\}\int_{-\infty}^{-c-a/2\epsilon^2} \exp\left\{-\epsilon^2 r^2\right\}\,\mathrm{d}r \\
&+ \frac{1}{2}\exp\left\{\frac{b^2}{4\epsilon^2}\right\}\int_{-\infty}^{-c-b/2\epsilon^2} \exp\left\{-\epsilon^2 s^2\right\}\,\mathrm{d}s \\
&+ \frac{1}{2}\exp\left\{\frac{a^2}{4\epsilon^2}\right\}\int_{c-a/2\epsilon^2}^{\infty} \exp\left\{-\epsilon^2 r^2\right\}\,\mathrm{d}r \\
&+ \frac{1}{2}\exp\left\{\frac{b^2}{4\epsilon^2}\right\}\int_{c-b/2\epsilon^2}^{\infty} \exp\left\{-\epsilon^2 s^2\right\}\,\mathrm{d}s. \qquad (5.55)
\end{aligned}
$$

The anti-derivative of $\exp\{-\beta^2 x^2\} = \frac{\sqrt{\pi}\mathrm{erf}(\beta x)}{2\beta}$, where erf is the error function.

Then equation (5.55) becomes:

$$\begin{aligned} f(t) &= \frac{1}{2}\exp\left\{\frac{a^2}{4\epsilon^2}\right\}\left[\frac{\sqrt{\pi}\mathrm{erf}(\epsilon r)}{2\epsilon}\right]_{-\infty}^{-c-a/2\epsilon^2} \\ &+ \frac{1}{2}\exp\left\{\frac{b^2}{4\epsilon^2}\right\}\left[\frac{\sqrt{\pi}\mathrm{erf}(\epsilon s)}{2\epsilon}\right]_{-\infty}^{-c-b/2\epsilon^2} \\ &+ \frac{1}{2}\exp\left\{\frac{a^2}{4\epsilon^2}\right\}\left[\frac{\sqrt{\pi}\mathrm{erf}(\epsilon r)}{2\epsilon}\right]_{c-a/2\epsilon^2}^{\infty} \\ &+ \frac{1}{2}\exp\left\{\frac{b^2}{4\epsilon^2}\right\}\left[\frac{\sqrt{\pi}\mathrm{erf}(\epsilon s)}{2\epsilon}\right]_{c-b/2\epsilon^2}^{\infty}. \end{aligned} \tag{5.56}$$

We introduce an important result from Herrmann [39], where he refers to the definition of the complex error function as seen in Abramowitz and Stegun [2]. Herrmann notes that the complex error function $\mathrm{erf}(x+iy)$ is an odd function of its argument and that $\mathrm{erf}(x+iy) \approx 1$ for $x > |y|$ and $x > 2$, and that $\mathrm{erf}(x+iy) \approx -1$ for $x < -|y|$ and $x < -2$.

We write equation (5.56) as:

$$\begin{aligned} f(t) &= \frac{\sqrt{\pi}}{4\epsilon}\exp\left\{\frac{a^2}{4\epsilon^2}\right\}\left[\mathrm{erf}\left(-c\epsilon - \frac{a}{2\epsilon}\right) - (-1)\right] + \frac{\sqrt{\pi}}{4\epsilon}\exp\left\{\frac{b^2}{4\epsilon^2}\right\}\left[\mathrm{erf}\left(-c\epsilon - \frac{b}{2\epsilon}\right) - (-1)\right] \\ &+ \frac{\sqrt{\pi}}{4\epsilon}\exp\left\{\frac{a^2}{4\epsilon^2}\right\}\left[1 - \mathrm{erf}\left(c\epsilon - \frac{a}{2\epsilon}\right)\right] + \frac{\sqrt{\pi}}{4\epsilon}\exp\left\{\frac{b^2}{4\epsilon^2}\right\}\left[1 - \mathrm{erf}\left(c\epsilon - \frac{b}{2\epsilon}\right)\right]. \end{aligned} \tag{5.57}$$

Collecting terms together:

$$
\begin{aligned}
f(t) &= \frac{\sqrt{\pi}}{4\epsilon}\exp\left\{\frac{a^2}{4\epsilon^2}\right\}\left[2+\operatorname{erf}\left(-c\epsilon-\frac{a}{2\epsilon}\right)-\operatorname{erf}\left(c\epsilon-\frac{a}{2\epsilon}\right)\right] \\
&+ \frac{\sqrt{\pi}}{4\epsilon}\exp\left\{\frac{b^2}{4\epsilon^2}\right\}\left[2+\operatorname{erf}\left(-c\epsilon-\frac{b}{2\epsilon}\right)-\operatorname{erf}\left(c\epsilon-\frac{b}{2\epsilon}\right)\right]. \qquad (5.58)
\end{aligned}
$$

Noting that the error function is an odd function we can express f as:

$$
\begin{aligned}
f(t) &= \frac{\sqrt{\pi}}{4\epsilon}\exp\left\{\frac{a^2}{4\epsilon^2}\right\}\left[1-\operatorname{erf}\left(c\epsilon+\frac{a}{2\epsilon}\right)\right]+\frac{\sqrt{\pi}}{4\epsilon}\exp\left\{\frac{a^2}{4\epsilon^2}\right\}\left[1-\operatorname{erf}\left(c\epsilon-\frac{a}{2\epsilon}\right)\right] \\
&+ \frac{\sqrt{\pi}}{4\epsilon}\exp\left\{\frac{b^2}{4\epsilon^2}\right\}\left[1-\operatorname{erf}\left(c\epsilon+\frac{b}{2\epsilon}\right)\right]+\frac{\sqrt{\pi}}{4\epsilon}\exp\left\{\frac{b^2}{4\epsilon^2}\right\}\left[1-\operatorname{erf}\left(c\epsilon-\frac{b}{2\epsilon}\right)\right] \\
&= \frac{\sqrt{\pi}}{4\epsilon}\left[T_1(t)+T_2(t)+T_3(t)+T_4(t)\right]. \qquad (5.59)
\end{aligned}
$$

We remind the reader of the definition of a Faddeeva Function that we introduced in section 5.2, equation (5.13), defined as:

$$
\begin{aligned}
w(z) &= \exp(-z^2)\operatorname{erfc}(-iz) = \exp(-z^2)\left[1-\operatorname{erf}(-iz)\right] \\
&= \exp(y^2-2ixy-x^2)\left[1-\operatorname{erf}(y-ix)\right],
\end{aligned}
$$

where erfc is the complementary error function and $z = x+iy \in \mathbb{C}$.

We can rearrange equation (5.59) such that we can write it as the sum of Faddeeva functions. We take each term in turn, noting that $a = \lambda + it$:

$$
\begin{aligned}
T_1(t) &= \exp\left\{\frac{a^2}{4\epsilon^2}\right\}\left[1-\operatorname{erf}\left(c\epsilon+\frac{a}{2\epsilon}\right)\right] \\
&= \exp\left\{\frac{(\lambda+it)^2}{4\epsilon^2}\right\}\left[1-\operatorname{erf}\left(c\epsilon+\frac{\lambda+it}{2\epsilon}\right)\right].
\end{aligned} \tag{5.60}
$$

Comparing the error term from equation (5.60) with that of the Faddeeva function, we want $z_1 = x_1 + iy_1$, where $x_1 = -t/2\epsilon$ and $y_1 = c\epsilon + \lambda/2\epsilon$. We expand the exponential function

$$
T_1(t) = \exp\left\{\frac{\lambda^2+2i\lambda t-t^2}{4\epsilon^2}\right\}\left[1-\operatorname{erf}\left(c\epsilon+\frac{\lambda+it}{2\epsilon}\right)\right], \tag{5.61}
$$

while

$$
\begin{aligned}
\exp\left\{-z_1^2\right\} &= \exp\left\{-\left(-\frac{t}{2\epsilon}+ic\epsilon+i\frac{\lambda}{2\epsilon}\right)^2\right\} \\
&= \exp\left\{\frac{\lambda^2+2i\lambda t-t^2}{4\epsilon^2}\right\}\exp\left\{itc+c^2\epsilon^2+c\lambda\right\}.
\end{aligned} \tag{5.62}
$$

Combining the above, it follows that

$$
\begin{aligned}
T_1(t) &= \exp\left\{-itc-c^2\epsilon^2-c\lambda\right\}\exp\left\{-z_1^2\right\}\left[1-\operatorname{erf}\left(c\epsilon+\frac{\lambda+it}{2\epsilon}\right)\right] \\
&= \exp\left\{-itc-c^2\epsilon^2-c\lambda\right\}w(z_1),
\end{aligned} \tag{5.63}
$$

where $w(z_1)$ is the Faddeeva function of $z_1 = -t/2\epsilon + iy_1$ where $y_1 = (c\epsilon + \lambda/2\epsilon)$.

We perform similar calculations for T_2:

$$\begin{aligned} T_2(t) &= \exp\left\{\frac{a^2}{4\epsilon^2}\right\}\left[1-\operatorname{erf}\left(c\epsilon-\frac{a}{2\epsilon}\right)\right] \\ &= \exp\left\{\frac{(\lambda+it)^2}{4\epsilon^2}\right\}\left[1-\operatorname{erf}\left(c\epsilon-\frac{(\lambda+it)}{2\epsilon}\right)\right]. \end{aligned} \tag{5.64}$$

Comparing the error term from equation (5.64) with that of the Faddeeva function, we want $z_2 = x_2 + iy_2$, where $x_2 = t/2\epsilon$ and $y_2 = c\epsilon - \lambda/2\epsilon$. We expand the exponential function

$$T_2(t) = \exp\left\{\frac{\lambda^2 + 2i\lambda t - t^2}{4\epsilon^2}\right\}\left[1-\operatorname{erf}\left(c\epsilon-\frac{(\lambda+it)}{2\epsilon}\right)\right], \tag{5.65}$$

while

$$\begin{aligned} \exp\left\{-z_2^2\right\} &= \exp\left\{-\left(\frac{t}{2\epsilon}+ic\epsilon-i\frac{\lambda}{2\epsilon}\right)^2\right\} \\ &= \exp\left\{\frac{\lambda^2 + 2i\lambda t - t^2}{4\epsilon^2}\right\}\exp\left\{-itc + c^2\epsilon^2 - c\lambda\right\}. \end{aligned} \tag{5.66}$$

Combining the above, it follows that

$$\begin{aligned} T_2(t) &= \exp\left\{itc - c^2\epsilon^2 + c\lambda\right\}\exp\left\{-z_2^2\right\}\left[1-\operatorname{erf}\left(c\epsilon-\frac{(\lambda+it)}{2\epsilon}\right)\right] \\ &= \exp\left\{itc - c^2\epsilon^2 + c\lambda\right\} w(z_2), \end{aligned} \tag{5.67}$$

where $w(z_2)$ is the Faddeeva function of $z_2 = t/2\epsilon + iy_2$ where $y_2 = (c\epsilon - \lambda/2\epsilon)$.

We perform similar calculations for T_3, noting that $b = it - \lambda$:

$$\begin{aligned} T_3(t) &= \exp\left\{\frac{b^2}{4\epsilon^2}\right\}\left[1 - \operatorname{erf}\left(c\epsilon + \frac{b}{2\epsilon}\right)\right] \\ &= \exp\left\{\frac{(it-\lambda)^2}{4\epsilon^2}\right\}\left[1 - \operatorname{erf}\left(c\epsilon + \frac{(it-\lambda)}{2\epsilon}\right)\right]. \end{aligned} \tag{5.68}$$

Comparing the error term from equation (5.68) with that of the Faddeeva function, we want $z_3 = x_3 + iy_3$, where $x_3 = -t/2\epsilon$ and $y_3 = c\epsilon - \lambda/2\epsilon$. We expand the exponential function

$$T_3(t) = \exp\left\{\frac{\lambda^2 - 2i\lambda t - t^2}{4\epsilon^2}\right\}\left[1 - \operatorname{erf}\left(c\epsilon + \frac{(it-\lambda)}{2\epsilon}\right)\right], \tag{5.69}$$

while

$$\begin{aligned} \exp\left\{-z_3^2\right\} &= \exp\left\{-\left(-\frac{t}{2\epsilon} + ic\epsilon - i\frac{\lambda}{2\epsilon}\right)^2\right\} \\ &= \exp\left\{\frac{\lambda^2 - 2i\lambda t - t^2}{4\epsilon^2}\right\}\exp\left\{itc + c^2\epsilon^2 - c\lambda\right\}. \end{aligned} \tag{5.70}$$

Combining the above, it follows that

$$\begin{aligned} T_3(t) &= \exp\left\{-itc - c^2\epsilon^2 + c\lambda\right\} \exp\left\{-z_3^2\right\}\left[1 - \operatorname{erf}\left(c\epsilon + \frac{(it-\lambda)}{2\epsilon}\right)\right] \\ &= \exp\left\{-itc - c^2\epsilon^2 + c\lambda\right\} w(z_3), \end{aligned} \tag{5.71}$$

where $w(z_3)$ is the Faddeeva function of $z_3 = -t/2\epsilon + iy_3$ where $y_3 = (c\epsilon - \lambda/2\epsilon)$.

Finally, for T_4:

$$\begin{aligned} T_4(t) &= \exp\left\{\frac{b^2}{4\epsilon^2}\right\}\left[1 - \operatorname{erf}\left(c\epsilon - \frac{b}{2\epsilon}\right)\right] \\ &= \exp\left\{\frac{(it-\lambda)^2}{4\epsilon^2}\right\}\left[1 - \operatorname{erf}\left(c\epsilon - \frac{(it-\lambda)}{2\epsilon}\right)\right]. \end{aligned} \tag{5.72}$$

Comparing the error term from equation (5.72) with that of the Faddeeva function, we want $z_4 = x_4 + iy_4$, where $x_4 = t/2\epsilon$ and $y_4 = c\epsilon + \lambda/2\epsilon$. We expand the exponential function

$$T_4(t) = \exp\left\{\frac{\lambda^2 - 2i\lambda t - t^2}{4\epsilon^2}\right\}\left[1 - \operatorname{erf}\left(c\epsilon - \frac{(it-\lambda)}{2\epsilon}\right)\right], \tag{5.73}$$

while

$$\begin{aligned} \exp\left\{-z_4^2\right\} &= \exp\left\{-\left(\frac{t}{2\epsilon} + ic\epsilon + i\frac{\lambda}{2\epsilon}\right)^2\right\} \\ &= \exp\left\{\frac{\lambda^2 - 2i\lambda t - t^2}{4\epsilon^2}\right\}\exp\left\{-itc + c^2\epsilon^2 + c\lambda\right\}. \end{aligned} \tag{5.74}$$

Combining the above, it follows that

$$\begin{aligned} T_4(t) &= \exp\left\{itc - c^2\epsilon^2 - c\lambda\right\} \exp\left\{-z_4^2\right\} \left[1 - \operatorname{erf}\left(c\epsilon - \frac{(it-\lambda)}{2\epsilon}\right)\right] \\ &= \exp\left\{itc - c^2\epsilon^2 - c\lambda\right\} w(z_4), \end{aligned} \tag{5.75}$$

where $w(z_4)$ is the Faddeeva function of $z_4 = t/2\epsilon + iy_4$ where $y_4 = (ce + \lambda/2\epsilon)$.

Combining the above, we can write equation (5.59) as:

$$\begin{aligned} f(t) &= \frac{\sqrt{\pi}}{4\epsilon}\left[T_1(t) + T_2(t) + T_3(t) + T_4(t)\right] \\ &= \frac{\sqrt{\pi}}{4\epsilon} \exp\left\{-c^2\epsilon^2\right\} \left[\exp\{-itc - c\lambda\} w(z_1) + \exp\{itc + c\lambda\} w(z_2)\right. \\ &+ \left.\exp\{-itc + c\lambda\} w(z_3) + \exp\{itc - c\lambda\} w(z_4)\right]. \end{aligned} \tag{5.76}$$

There are tables for values of the Faddeeva function $w(x+iy)$ for positive x and positive y (Abramowitz and Stegun [2], pages 325-328). Let us assume that we can choose c and ϵ, such that $2c\epsilon^2 > \lambda$; then our imaginary part will always be positive. Finally we note that for negative real part, which we have seen in $w(z_1)$ and $w(z_2)$, we can write $w(-x+iy) = \overline{w(x+iy)}$. Hence, we can express the following four Faddeeva functions as:

$$
\begin{aligned}
w(z_1) &= w\left(\frac{-t}{2\epsilon} + i\left[c\epsilon + \frac{\lambda}{2\epsilon}\right]\right) = \overline{w\left(\frac{t}{2\epsilon} + i\left[c\epsilon + \frac{\lambda}{2\epsilon}\right]\right)} = \overline{w(z_4)}, \\
w(z_2) &= w\left(\frac{t}{2\epsilon} + i\left[c\epsilon - \frac{\lambda}{2\epsilon}\right]\right), \\
w(z_3) &= w\left(\frac{-t}{2\epsilon} + i\left[c\epsilon - \frac{\lambda}{2\epsilon}\right]\right) = \overline{w\left(\frac{t}{2\epsilon} + i\left[c\epsilon - \frac{\lambda}{2\epsilon}\right]\right)} = \overline{w(z_2)}, \\
w(z_4) &= w\left(\frac{t}{2\epsilon} + i\left[c\epsilon + \frac{\lambda}{2\epsilon}\right]\right).
\end{aligned}
$$

Substituting these back into equation (5.76):

$$
\begin{aligned}
f(t) &= \frac{\sqrt{\pi}}{4\epsilon}\exp\left\{-c^2\epsilon^2\right\}\Big[\exp\{-itc - c\lambda\}\overline{w(z_4)} + \exp\{itc + c\lambda\}w(z_2) \\
&+ \exp\{-itc + c\lambda\}\overline{w(z_2)} + \exp\{itc - c\lambda\}w(z_4)\Big] \\
&= \frac{\sqrt{\pi}}{4\epsilon}\exp\left\{-c^2\epsilon^2\right\}\Big[e^{-c\lambda}(\cos(ct) - i\sin(ct))\overline{w(z_4)} + e^{-c\lambda}(\cos(ct) + i\sin(ct))w(z_4) \\
&+ e^{c\lambda}(\cos(ct) - i\sin(ct))\overline{w(z_2)} + e^{c\lambda}(\cos(ct) + i\sin(ct))w(z_2)\Big] \\
&= \frac{\sqrt{\pi}}{4\epsilon}\exp\left\{-c^2\epsilon^2\right\}\Big[e^{-c\lambda}(2\cos(ct)\Re(w(z_4)) - 2\sin(ct)\Im(w(z_4))) \\
&+ e^{c\lambda}(2\cos(ct)\Re(w(z_2)) - 2\sin(ct)\Im(w(z_2)))\Big] \\
&= \frac{\sqrt{\pi}}{2\epsilon}\exp\left\{-c^2\epsilon^2\right\}\Big[e^{-c\lambda}(\cos(ct)\Re(w(z_4)) - \sin(ct)\Im(w(z_4))) \\
&+ e^{c\lambda}(\cos(ct)\Re(w(z_2)) - \sin(ct)\Im(w(z_2)))\Big]. \qquad (5.77)
\end{aligned}
$$

Referring to the tables of Abramowitz and Stegun [2], we can see that the real and imaginary parts of $w(z_2) = w(t/2\epsilon + iy_2)$ and $w(z_4) = w(t/2\epsilon + iy_4)$ are always less than or equal to 1. In fact, as t increases, eventually the real and imaginary parts of $w(z_2) = w(t/2\epsilon + iy_2)$ and $w(z_4) = w(t/2\epsilon + iy_4)$ become insignificant.

The tables only show values for the real and imaginary parts of the Faddeeva function $w(x+iy)$ for $x \leq 3.9$ and $y \leq 3$. However, there is an expression for $w(x+iy)$ for larger values of x and y:

$$w(z) = iz\left(\frac{A_1}{z^2 - A_2} + \frac{A_3}{z^2 - A_4} + \frac{A_5}{z^2 - A_6}\right) + \eta(z), \qquad (5.78)$$

where:

$$\begin{aligned} A_1 &= 0.4613135,\ A_2 = 0.1901635,\ A_3 = 0.09999216 \\ A_4 &= 1.7844927,\ A_5 = 0.002883894,\ A_6 = 5.5253437 \end{aligned}$$

and $|\eta(z)| < 2 \times 10^{-6}$.

We would like to demonstrate that the function f becomes insignificant outside some closed interval, i.e. $|f(t)| \leq M(\epsilon, \lambda, c)$ for $t \in \mathbb{R} \setminus [-b, b]$, where $M(\epsilon, \lambda, c)$ is a constant that depends on the values: λ, ϵ and c.

We calculate $|\Re[w(z_2)]|$ and $|\Im[w(z_2)]|$. We use equation (5.76):

$$
\begin{aligned}
w\left(z_2\right) &= iz_2\left(\frac{A_1}{z_2^2-A_2}+\frac{A_3}{z_2^2-A_4}+\frac{A_5}{z_2^2-A_6}\right)\\
&= iz_2\left(\frac{A_1\left(\overline{z}_2^2-A_2\right)}{\left(z_2^2-A_2\right)\left(\overline{z}_2^2-A_2\right)}+\frac{A_3\left(\overline{z}_2^2-A_4\right)}{\left(z_2^2-A_4\right)\left(\overline{z}_2^2-A_4\right)}+\frac{A_5\left(\overline{z}_2^2-A_6\right)}{\left(z_2^2-A_6\right)\left(\overline{z}_2^2-A_6\right)}\right)\\
&= i\left(x_2+iy_2\right)\left(\frac{A_1\left(\overline{z}_2^2-A_2\right)}{\left(x_2^2y_2^2-A_2\right)^2+4x_2^2y_2^2}+\frac{A_3\left(\overline{z}_2^2-A_4\right)}{\left(x_2^2y_2^2-A_4\right)^2+4x_2^2y_2^2}\right.\\
&+ \left.\frac{A_5\left(\overline{z}_2^2-A_6\right)}{\left(x_2^2y_2^2-A_6\right)^2+4x_2^2y_2^2}\right), \qquad (5.79)
\end{aligned}
$$

where $\overline{z}_2 = x_2 - iy_2$ is the complex conjugate of $z = x_2 + iy_2$.

Now,

$$
\overline{z}_2^2 = \left(x_2 - iy_2\right)^2 = x_2^2 - 2ix_2y_2 - y_2^2. \qquad (5.80)
$$

Substituting this into equation (5.79), we have:

$$
\begin{aligned}
w\left(z_2\right) &= i\left(x_2+iy_2\right)\left(\frac{A_1\left(x_2^2-y_2^2-A_2-2ix_2y_2\right)}{\left(x_2^2y_2^2-A_2\right)^2+4x_2^2y_2^2}+\frac{A_3\left(x_2^2-y_2^2-A_4-2ix_2y_2\right)}{\left(x_2^2y_2^2-A_4\right)^2+4x_2^2y_2^2}\right.\\
&+ \left.\frac{A_5\left(x_2^2-y_2^2-A_6-2ix_2y_2\right)}{\left(x_2^2y_2^2-A_6\right)^2+4x_2^2y_2^2}\right). \qquad (5.81)
\end{aligned}
$$

Noting that $z_2 = x/2 + i\alpha$, where $x = t/2\epsilon$ and $\alpha = c\epsilon - \lambda/2\epsilon$. We can write equation (5.81) as

$$\begin{aligned}
w(z_2) &= (ix-\alpha)\left(\frac{A_1\left(x^2-\alpha^2-A_2-2i\alpha x\right)}{\left(\alpha^2x^2-A_2\right)^2+4\alpha^2x^2}+\frac{A_3\left(x^2-\alpha^2-A_4-2i\alpha x\right)}{\left(\alpha^2x^2-A_4\right)^2+4\alpha^2x^2}\right. \\
&+ \left.\frac{A_5\left(x^2-\alpha^2-A_6-2i\alpha x\right)}{\left(\alpha^2x^2-A_6\right)^2+4\alpha^2x^2}\right) \\
&= \frac{A_1\left[\alpha x^2+\alpha^3+\alpha A_2+i\left(x^3+\alpha^2x-A_2x\right)\right]}{\left(\alpha^2x^2-A_2\right)^2+4\alpha^2x^2} \\
&+ \frac{A_3\left[\alpha x^2+\alpha^3+\alpha A_4+i\left(x^3+\alpha^2x-A_4x\right)\right]}{\left(\alpha^2x^2-A_4\right)^2+4\alpha^2x^2} \\
&+ \frac{A_5\left[\alpha x^2+\alpha^3+\alpha A_6+i\left(x^3+\alpha^2x-A_6x\right)\right]}{\left(\alpha^2x^2-A_6\right)^2+4\alpha^2x^2}.
\end{aligned} \tag{5.82}$$

Splitting equation (5.82) into its real and imaginary parts, we have:

$$\Re\left[w(z_2)\right] = \frac{A_1\left[\alpha x^2+\alpha^3+\alpha A_2\right]}{\left(\alpha^2x^2-A_2\right)^2+4\alpha^2x^2}+\frac{A_3\left[\alpha x^2+\alpha^3+\alpha A_4\right]}{\left(\alpha^2x^2-A_4\right)^2+4\alpha^2x^2}+\frac{A_5\left[\alpha x^2+\alpha^3+\alpha A_6\right]}{\left(\alpha^2x^2-A_6\right)^2+4\alpha^2x^2}, \tag{5.83}$$

$$\Im\left[w(z_2)\right] = \frac{A_1\left[x^3+\alpha^2x-A_2x\right]}{\left(\alpha^2x^2-A_2\right)^2+4\alpha^2x^2}+\frac{A_3\left[x^3+\alpha^2x-A_4x\right]}{\left(\alpha^2x^2-A_4\right)^2+4\alpha^2x^2}+\frac{A_5\left[x^3+\alpha^2x-A_6x\right]}{\left(\alpha^2x^2-A_6\right)^2+4\alpha^2x^2}. \tag{5.84}$$

Taking absolute values, for $\Re\left[w(z_2)\right]$ we have:

$$\left|\Re\left[w(z_2)\right]\right| = \left|\frac{A_1\left[\alpha x^2+\alpha^3+\alpha A_2\right]}{\left(\alpha^2x^2-A_2\right)^2+4\alpha^2x^2}+\frac{A_3\left[\alpha x^2+\alpha^3+\alpha A_4\right]}{\left(\alpha^2x^2-A_4\right)^2+4\alpha^2x^2}+\frac{A_5\left[\alpha x^2+\alpha^3+\alpha A_6\right]}{\left(\alpha^2x^2-A_6\right)^2+4\alpha^2x^2}\right|. \tag{5.85}$$

Let $B_1 = \alpha^2 + A_2$, $B_2 = \alpha^2 + A_4$ and $B_3 = \alpha^2 + A_6$, then we can express equation (5.85) as:

$$|\Re\left[w\left(z_2\right)\right]| \quad = \quad \left|\frac{\alpha A_1\left[x^2 + B_1\right]}{\left(\alpha^2 x^2 - A_2\right)^2 + 4\alpha^2 x^2} + \frac{\alpha A_3\left[x^2 + B_2\right]}{\left(\alpha^2 x^2 - A_4\right)^2 + 4\alpha^2 x^2} + \frac{\alpha A_5\left[x^2 + B_3\right]}{\left(\alpha^2 x^2 - A_6\right)^2 + 4\alpha^2 x^2}\right|. \tag{5.86}$$

Let $u = x^2$, and if we bring the expression over a common denominator, the dominator becomes:

$$\left(\left(\alpha^2 u - A_2\right)^2 + 4\alpha^2 u\right)\left(\left(\alpha^2 u - A_4\right)^2 + 4\alpha^2 u\right)\left(\left(\alpha^2 u - A_6\right)^2 + 4\alpha^2 u\right), \tag{5.87}$$

while the numerator can be expressed as:

$$\begin{aligned} N_1 + N_2 + N_3 \quad = \quad & \alpha A_1\left[u + B_1\right]\left[\left(\alpha^2 u - A_4\right)^2 + 4\alpha^2 u\right]\left[\left(\alpha^2 u - A_6\right)^2 + 4\alpha^2 u\right] \\ + \quad & \alpha A_3\left[u + B_2\right]\left[\left(\alpha^2 u - A_2\right)^2 + 4\alpha^2 u\right]\left[\left(\alpha^2 u - A_6\right)^2 + 4\alpha^2 u\right] \\ + \quad & \alpha A_5\left[u + B_3\right]\left[\left(\alpha^2 u - A_2\right)^2 + 4\alpha^2 u\right]\left[\left(\alpha^2 u - A_4\right)^2 + 4\alpha^2 u\right]. \end{aligned} \tag{5.88}$$

Considering each term in turn, we have:

$$
\begin{aligned}
N_1 &= \alpha A_1 \left[u + B_1\right] \left[\left(\alpha^2 u - A_4\right)^2 + 4\alpha^2 u\right] \left[\left(\alpha^2 u - A_6\right)^2 + 4\alpha^2 u\right] && (5.89) \\
&= \alpha A_1 \left[u + B_1\right] \left[\alpha^8 u^4 + C_1 \alpha^6 u^3 + C_2 \alpha^4 u^2 + C_3 \alpha^2 u + \alpha C_4\right], && (5.90)
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &= -6.6196728 = 2\left(4 - A_4 - A_6\right), \\
C_2 &= 1.095081497 = A_6^2 + A_4^2 + 16 - 8A_4 - 8A_6 + A_4 A_6, \\
C_3 &= -9.293682046 = 4A_6^2 + 4A_4^2 - 2A_4 A_6^2 - 2A_4^2 A_6, \\
C_4 &= 97.221832802 = A_4^2 A_6^2.
\end{aligned}
$$

Since N_1, N_2 and N_3 are equal except for different constants we can deduce that:

$$
\begin{aligned}
N_2 &= \alpha A_3 \left[u + B_2\right] \left[\alpha^8 u^4 + D_1 \alpha^6 u^3 + D_2 \alpha^4 u^2 + D_3 \alpha^2 u + \alpha D_4\right], \\
N_3 &= \alpha A_5 \left[u + B_3\right] \left[\alpha^8 u^4 + E_1 \alpha^6 u^3 + E_2 \alpha^4 u^2 + E_3 \alpha^2 u + \alpha E_4\right], && (5.91)
\end{aligned}
$$

where

$$
\begin{aligned}
D_1 &= -3.4310144 = 2\left(4 - A_2 - A_6\right), \\
D_2 &= 1.892246257 = A_6^2 + A_2^2 + 16 - 8A_2 - 8A_6 + A_2A_6, \\
D_3 &= 110.2515601 = 4A_6^2 + 4A_2^2 - 2A_2A_6^2 - 2A_2^2A_6, \\
D_4 &= 1.10400978 = A_2^2A_6^2,
\end{aligned}
$$

and

$$
\begin{aligned}
E_1 &= 4.0506876 = 2\left(4 - A_2 - A_4\right), \\
E_2 &= 3.762672131 = A_2^2 + A_4^2 + 16 - 8A_2 - 8A_4 + A_2A_4, \\
E_3 &= 12.62418099 = 4A_2^2 + 4A_4^2 - 2A_2A_4^2 - 2A_2^2A_4, \\
E_4 &= 0.115155285 = A_2^2A_4^2.
\end{aligned}
$$

Combining the above, the numerator of equation (5.84) becomes:

$$
\begin{aligned}
N_1 + N_2 + N_3 &= \alpha A_1\left[u + B_1\right]\left[\alpha^8u^4 + C_1\alpha^6u^3 + C_2\alpha^4u^2 + C_3\alpha^2u + \alpha C_4\right] \\
&+ \alpha A_3\left[u + B_2\right]\left[\alpha^8u^4 + D_1\alpha^6u^3 + D_2\alpha^4u^2 + D_3\alpha^2u + \alpha D_4\right] \\
&+ \alpha A_5\left[u + B_3\right]\left[\alpha^8u^4 + E_1\alpha^6u^3 + E_2\alpha^4u^2 + E_3\alpha^2u + \alpha E_4\right]. \quad (5.92)
\end{aligned}
$$

Bringing terms together we can express the numerator as:

$$\theta_1\alpha^9u^5 + \theta_2\alpha^7u^4 + \theta_3\alpha^5u^3 + \theta_4\alpha^3u^2 + \theta_5\alpha u + \theta_6 \tag{5.93}$$

where

$$\begin{aligned}
\theta_1 &= 0.75146916 = A_1 + A_2 + A_3,\\
\theta_2 &= -3.385137214 + 0.090568268\alpha^2 + 0.564189554\alpha^4\\
&= A_1C_1 + A_3D_1 + A_5E_1 + (A_1A_2 + A_3A_4 + A_5A_6)\,\alpha^2 + (A_1 + A_3 + A_5)\,\alpha^4,\\
\theta_3 &= 0.705236816 - 1.128379038\alpha^2 - 3.385137215\alpha^4\\
&= A_1C_2 + A_3D_2 + A_5E_2 + (A_1C_1A_2 + A_3D_1A_4 + A_5E_1A_6)\,\alpha^2 + (A_1C_1 + A_3D_1 + A_5E_1)\,\alpha^4,\\
\theta_4 &= 6.773397462 + 0.493665823\alpha^2 + 0.705236816\alpha^4\\
&= A_1C_3 + A_3D_3 + A_5E_3 + (A_1C_2A_2 + A_3D_2A_4 + A_5E_2A_6)\,\alpha^2 + (A_1C_2 + A_3D_2 + A_5E_2)\,\alpha^4,\\
\theta_5 &= 44.95885158 + 19.05863987\alpha^2 + 6.773397462\alpha^4\\
&= A_1C_4 + A_3D_4 + A_5E_4 + (A_1C_3A_2 + A_3D_3A_4 + A_5E_3A_6)\,\alpha^2 + (A_1C_3 + A_3D_3 + A_5E_3)\,\alpha^4,\\
\theta_6 &= 8.727306066\alpha^2 + 44.95885158\alpha^4\\
&= (A_1C_4A_2 + A_3D_4A_4 + A_5E_4A_6)\,\alpha^2 + (A_1C_4 + A_3D_4 + A_5E_4)\,\alpha^4.
\end{aligned}$$

We evaluate the denominator, to give:

$$\alpha^{12}u^6 + \sigma_1\alpha^{10}u^5 + \sigma_2\alpha^8u^4 + 2\sigma_3\alpha^6u^3 + \sigma_4\alpha^4u^2 + 2\sigma_5\alpha^2u + \sigma_6, \tag{5.94}$$

where

$$
\begin{aligned}
\sigma_1 &= -1.4999999 = 6 - A_2 - A_4 - A_6, \\
\sigma_2 &= 6.749999244 = A_2^2 + A_4^2 + A_6^2 + 48 - 16\left(A_2 + A_4 + A_6\right) + 4\left(A_2A_4 + A_2A_6 + A_4A_6\right), \\
\sigma_3 &= 50.75000009 = 32 - 16\left(A_2 + A_4 + A_6\right) + 4\left(A_2^2 + A_4^2 + A_6^2\right) + 8\left(A_2A_4 + A_2A_6 + A_4A_6\right) \\
&- 4A_2A_4A_6 - A_2^2\left(A_4 + A_6\right) - A_4^2\left(A_2 + A_6\right) - A_6^2\left(A_2 + A_4\right), \\
\sigma_4 &= 64.68750829 = A_2^2A_4^2 + A_2^2A_6^2 + A_4^2A_6^2 + 16\left(A_2^2 + A_4^2 + A_6^2\right) + 4\left(A_2A_4A_6^2 + A_2A_4^2A_6 + A_2^2A_4A_6\right) \\
&- 8A_2^2\left(A_4 + A_6\right) - 8A_4^2\left(A_2 + A_6\right) - 8A_6^2\left(A_2 + A_4\right), \\
\sigma_5 &= 175.7812387 = 2A_2^2A_4^2 + 2A_2^2A_6^2 + 2A_4^2A_6^2 - A_2A_4^2A_6^2 - A_2^2A_4A_6^2 - A_2^2A_4^2A_6, \\
\sigma_6 &= 3.515624415 = \left(A_2A_4A_6\right)^2.
\end{aligned}
$$

Combining the above, we have:

$$
\left|\Re\left[w\left(z_2\right)\right]\right| = \left|\frac{\theta_1\alpha^9u^5 + \theta_2\alpha^7u^4 + \theta_3\alpha^5u^3 + \theta_4\alpha^3u^2 + \theta_5\alpha u + \theta_6}{\alpha^{12}u^6 + \sigma_1\alpha^{10}u^5 + \sigma_2\alpha^8u^4 + 2\sigma_3\alpha^6u^3 + \sigma_4\alpha^4u^2 + 2\sigma_5\alpha^2u + \sigma_6}\right|. \quad (5.95)
$$

We note that the denominator is a polynomial of order 6 while the numerator is a polynomial of order 5. We can deduce that the ratio of the two polynomials will decrease as $|u| \to \infty$. Since $u = t^2/4\epsilon^2$ the decay is like $1/t^2$. It follows that equation (5.95) becomes very small for u outside some interval $[r, s]$ say, and hence for t outside $[2\epsilon\sqrt{r}, 2\epsilon\sqrt{s}]$. If we know the function g, then we also know what value α takes and hence we could solve $\left|\Re\left[w\left(z_2\right)\right]\right| \leq K_2$ for any value of K_2 we wish.

Next we consider the imaginary part of $w(z_2)$, we refer the reader to equation (5.84):

$$\Im\left[w\left(z_2\right)\right] = \frac{A_1\left[x^3+\alpha^2 x - A_2 x\right]}{\left(\alpha^2 x^2 - A_2\right)^2 + 4\alpha^2 x^2} + \frac{A_3\left[x^3+\alpha^2 x - A_4 x\right]}{\left(\alpha^2 x^2 - A_4\right)^2 + 4\alpha^2 x^2} + \frac{A_5\left[x^3+\alpha^2 x - A_6 x\right]}{\left(\alpha^2 x^2 - A_6\right)^2 + 4\alpha^2 x^2}. \tag{5.96}$$

If we bring the terms over a common denominator, we can see that we will have the same denominator as we had in the previous case, i.e. see equation (1.39) and equation (1.45). Hence we need only to evaluate the numerator;

$$\begin{aligned} N_4 + N_5 + N_6 &= A_1\left[x^3+\alpha^2 x - A_2 x\right]\left[\left(\alpha^2 x^2 - A_4\right)^2 + 4\alpha^2 x^2\right]\left[\left(\alpha^2 x^2 - A_6\right)^2 + 4\alpha^2 x^2\right] \\ &+ A_3\left[x^3+\alpha^2 x - A_4 x\right]\left[\left(\alpha^2 x^2 - A_2\right)^2 + 4\alpha^2 x^2\right]\left[\left(\alpha^2 x^2 - A_6\right)^2 + 4\alpha^2 x^2\right] \\ &+ A_5\left[x^3+\alpha^2 x - A_6 x\right]\left[\left(\alpha^2 x^2 - A_2\right)^2 + 4\alpha^2 x^2\right]\left[\left(\alpha^2 x^2 - A_4\right)^2 + 4\alpha^2 x^2\right]. \end{aligned} \tag{5.97}$$

We solve for N_1 and note that the only differences between N_4, N_5 and N_6 are different coefficients.

$$\begin{aligned} N_4 &= A_1\left[x^3+\alpha^2 x - A_2 x\right]\left[\left(\alpha^2 x^2 - A_4\right)^2 + 4\alpha^2 x^2\right]\left[\left(\alpha^2 x^2 - A_6\right)^2 + 4\alpha^2 x^2\right] \\ &= A_1\alpha^8 x^{11} + Q_1\alpha^6 x^9 + Q_2\alpha^4 x^7 + Q_3\alpha^2 x^5 + Q_4 x^3 + Q_5 x, \end{aligned} \tag{5.98}$$

where:

$$
\begin{aligned}
Q_1 &= 8A_1 - 2A_1A_6 - 2A_4A_1 - A_1A_2\alpha^2 + A_1\alpha^4, \\
Q_2 &= 16A_1 - 8A_1A_4 - 8A_1A_6 + 4A_1A_4A_6 + A_1A_4^2 + A_1A_6^2 \\
&+ 8A_1\alpha^4 - 2A_1A_6\alpha^4 - 2A_1A_4\alpha^4 - 8A_1A_2\alpha^2 + 2A_6A_1A_2\alpha^2 + 2A_4A_1A_2\alpha^2, \\
Q_3 &= 4A_1\left(A_4^2 + A_6^2\right) - 2A_1\left(A_4A_6^2 + A_4^2A_6\right) + 16A_1\alpha^4 - 8A_1\alpha^2\left(A_4\alpha^2 + A_6\alpha^2 - A_2A_4 - A_2A_6\right) \\
&+ A_1\alpha^4\left(A_4^2 + A_6^2\right) - A_1A_2\alpha^2\left(A_4^2 + A_6^2\right) - 16A_1A_2\alpha^2 + 4A_1A_4A_6\alpha^2\left(\alpha^2 - A_2\right), \\
Q_4 &= A_1A_4^2A_6^2 + 4A_1\alpha^4\left(A_6^2 + A_4^2\right) - 2A_1\alpha^4\left(A_4A_6^2 + A_4^2A_6\right) \\
&- 4A_1A_2\alpha^2\left(A_6^2 + A_4^2\right) + 2A_1A_2\left(A_4A_6^2 + A_4^2A_6\right), \\
Q_5 &= A_4^2A_6^2A_1\alpha^2 - A_1A_2A_4^2A_6^2.
\end{aligned}
$$

Comparing the coefficients of N_5 and N_6 with those of N_4, we can deduce that:

$$
N_4 + N_5 + N_6 = \rho_1\alpha^8x^{11} + \rho_2\alpha^6x^9 + \rho_3\alpha^4x^7 + \rho_4\alpha^2x^5 + \rho_5x^3 + \rho_6x, \quad (5.99)
$$

where:

$$
\begin{aligned}
\rho_1 &= 0.564189554 = A_1 + A_3 + A_5, \\
\rho_2 &= -3.385137215 - 0.282094775\alpha^2 + 0.564189554\alpha^4 \\
&= 8\left(A_1 + A_3 + A_5\right) - 2A_1\left(A_6 + A_4\right) - 2A_3\left(A_2 + A_6\right) - 2A_5\left(A_2 + A_4\right) \\
&- A_1A_2\alpha^2 - A_3A_4\alpha^2 - A_5A_6\alpha^2 + \alpha^4\left(A_1 + A_3 + A_5\right), \\
\rho_3 &= 14.66892768 + 1.128379038\alpha^2 - 3.385137215\alpha^4 \\
&= 16\left(A_1 + A_3 + A_5\right) - 8A_1\left(A_4 + A_6\right) - 8A_3\left(A_2 + A_6\right) - 8A_5\left(A_2 + A_4\right) \\
&+ 4A_1A_4A_6 + 4A_3A_2A_6 + 4A_5A_2A_4 + A_1\left(A_4^2 + A_6^2\right) + A_3\left(A_2^2 + A_6^2\right) + A_5\left(A_2^2 + A_4^2\right) \\
&+ 8\alpha^4\left(A_1 + A_3 + A_5\right) - 2A_1\alpha^4\left(A_6 + A_4\right) - 2A_3\alpha^4\left(A_2 + A_6\right) - 2A_5\alpha^4\left(A_2 + A_4\right) \\
&- 8\alpha^2\left(A_1A_2 + A_3A_4 + A_5A_6\right) + 2\alpha^2A_1A_2\left(A_6 + A_4\right) + 2\alpha^2A_3A_4\left(A_2 + A_6\right) + 2\alpha^2A_5A_6\left(A_2 + A_4\right), \\
\rho_4 &= 6.778006805 - 3.6672318\alpha^2 + 14.66892768\alpha^4 = 4A_1\left(A_4^2 + A_6^2\right) + 4A_3\left(A_2^2 + A_6^2\right) + 4A_5\left(A_2^2 + A_4^2\right) \\
&- 2A_1\left(A_4A_6^2 + A_4^2A_6\right) - 2A_3\left(A_2A_6^2 + A_2^2A_6\right) - 2A_5\left(A_2A_4^2 + A_2^2A_4\right) \\
&- 8A_1\alpha^2\left(A_4\alpha^2 + A_6\alpha^2 - A_2A_4 - A_2A_6\right) - 8A_3\alpha^2\left(A_2\alpha^2 + A_6\alpha^2 - A_4A_2 - A_4A_6\right) \\
&- 8A_5\alpha^2\left(A_2\alpha^2 + A_4\alpha^2 - A_6A_2 - A_6A_4\right) + A_1\alpha^4\left(A_4^2 + A_6^2\right) + A_3\alpha^4\left(A_2^2 + A_6^2\right) + A_5\alpha^4\left(A_2^2 + A_4^2\right) \\
&- A_1A_2\alpha^2\left(A_4^2 + A_6^2\right) - A_3A_4\alpha^2\left(A_2^2 + A_6^2\right) - A_5A_6\alpha^2\left(A_2^2 + A_4^2\right) \\
&+ 4A_1A_4A_6\alpha^2\left(\alpha^2 - A_2\right) + 4A_3A_2A_6\alpha^2\left(\alpha^2 - A_4\right) + 4A_5A_2A_4\alpha^2\left(\alpha^2 - A_6\right) \\
&- 16\alpha^2\left(A_1A_2 + A_3A_4 + A_5A_6\right) + 16\alpha^4\left(A_1 + A_3 + A_5\right), \\
\rho_5 &= 59.76882594 - 33.85137219\alpha^2 + 6.770276926\alpha^4 = A_1A_4^2A_6^2 + A_3A_2^2A_6^2 + A_5A_2^2A_4^2 \\
&+ 4A_1\alpha^4\left(A_6^2 + A_4^2\right) + 4A_3\alpha^4\left(A_2^2 + A_6^2\right) + 4A_5\alpha^4\left(A_2^2 + A_4^2\right) \\
&- 2A_1\alpha^4\left(A_4A_6^2 + A_4^2A_6\right) - 2A_3\alpha^4\left(A_2A_6^2 + A_2^2A_6\right) - 2A_5\alpha^4\left(A_2A_4^2 + A_2^2A_4\right) \\
&- 4A_1A_2\alpha^2\left(A_6^2 + A_4^2\right) - 4A_3A_4\alpha^2\left(A_2^2 + A_6^2\right) - 4A_5A_6\alpha^2\left(A_2^2 + A_4^2\right) \\
&+ 2A_1A_2\left(A_4A_6^2 + A_4^2A_6\right) + 2A_3A_4\left(A_2A_6^2 + A_2^2A_6\right) + 2A_5A_6\left(A_2A_4^2 + A_2^2A_4\right), \\
\rho_6 &= -8.727306066 + 44.95885158\alpha^2 \\
&= \alpha^2\left(A_4^2A_6^2A_1 + A_2^2A_6^2A_3 + A_2^2A_4^2A_5\right) - A_1A_2A_4^2A_6^2 - A_3A_4A_2^2A_6^2 - A_5A_6A_2^2A_4^2.
\end{aligned}
$$

Combining the above, we have:

$$|\Im\left[w\left(z_2\right)\right]| = \left|\frac{\rho_1\alpha^8x^{11} + \rho_2\alpha^6x^9 + \rho_3\alpha^4x^7 + \rho_4\alpha^2x^5 + \rho_5x^3 + \rho_6x}{\alpha^{12}x^{12} + \sigma_1\alpha^{10}x^{10} + \sigma_2\alpha^8x^8 + 2\sigma_3\alpha^6x^6 + \sigma_4\alpha^4x^4 + 2\sigma_5\alpha^2u + \sigma_6}\right|. \tag{5.100}$$

We note that the denominator is a polynomial of order 12 while the numerator is a polynomial of order 11. We can deduce that the ratio of the two polynomials will decrease as $|x| \to \infty$. Since $x = t/2\epsilon$ the decay is like $1/t$. It follows that equation (5.100) becomes very small for x outside some interval $[p, q]$ say, and hence for t outside $[2\epsilon p, 2\epsilon q]$. If we know the function g, then we also know what value α takes and hence we could solve $|\Im\left[w\left(z_2\right)\right]| \leq L_2$ for any value of L_2 we wish.

Now we demonstrate similar results using z_4. We note that the only difference between z_2 and z_4 is that we replace $\alpha = c\epsilon - \lambda/2\epsilon$ in the equations above with β where $\beta = c\epsilon + \lambda/2\epsilon$.

Hence, if we refer back to equation (5.77), we have shown that we can find an interval of compact essential numerical support for $f\left(t\right)$ by considering the values for which $|f\left(t\right)| \leq \delta$.

$$\begin{aligned}
|f(t)| &= \frac{\sqrt{\pi}}{2\epsilon}\exp\left\{-c^2\epsilon^2\right\}\left|\left[e^{-c\lambda}\left(\cos(ct)\,\Re(w(z_4)) - \sin(ct)\,\Im(w(z_4))\right)\right.\right.\\
&+ \left.\left. e^{c\lambda}\left(\cos(ct)\,\Re(w(z_2)) - \sin(ct)\,\Im(w(z_2))\right)\right]\right|\\
&\leq \frac{\sqrt{\pi}}{2\epsilon}\exp\left\{-c^2\epsilon^2\right\}\left\{e^{-c\lambda}\left|\cos(ct)\,\Re(w(z_4))\right| + e^{-c\lambda}\left|\sin(ct)\,\Im(w(z_4))\right|\right.\\
&+ \left. e^{c\lambda}\left|\cos(ct)\,\Re(w(z_2))\right| + e^{c\lambda}\left|\sin(ct)\,\Im(w(z_2))\right|\right\}\\
&\leq \frac{\sqrt{\pi}}{2\epsilon}\exp\left\{-c^2\epsilon^2\right\}\left\{e^{-c\lambda}\left|\Re(w(z_4))\right| + e^{-c\lambda}\left|\Im(w(z_4))\right|\right.\\
&+ \left. e^{c\lambda}\left|\Re(w(z_2))\right| + e^{c\lambda}\left|\Im(w(z_2))\right|\right\}. \qquad (5.101)
\end{aligned}$$

If we examine inequality (5.101), we note that there are competing terms on the right hand side. Outside the curly brackets, the terms $1/\epsilon$ and $\exp\{-c^2\epsilon^2\}$ have different behaviour as ϵ approaches zero. The absolute value of the real and imaginary parts of the Faddeeva function are bounded above by one. For the estimate to be useful for a large class of functions, we don't want c to be too small; if c is negligible, essentially we are only considering the case of $\hat{g}$ (and g) being Gaussians. We will be assuming that ϵ is relatively small and c relatively large; recalling that we have decomposed $h = j + f$, in this case $|f|$ would make a significant contribution to $|h|$. For many relationships between c and ϵ the $\exp\{-c^2\epsilon^2\}$ term will dominate; we can use (5.101) to ensure that $|f(t)|$ is small.

Now, if we refer back to the expression for the relaxation spectrum:

$$
\begin{aligned}
h(t) &= f(t) + \int_{-c}^{c} \hat{g}(x) \cosh(\lambda x) e^{itx} \, \mathrm{d}x \\
&= f(t) + j(t) \,.
\end{aligned} \tag{5.102}
$$

We would like to demonstrate that the relaxation spectrum has compact essential numerical support. We have demonstrated above that the function $f(t)$ does have compact essential numerical support, hence we need only consider the following expression for the remainder of the calculation:

$$
|j(t)| = \left| \int_{-c}^{c} \hat{g}(x) \cosh(\lambda x) e^{itx} \, \mathrm{d}x \right| . \tag{5.103}
$$

It is not obvious how to obtain the best estimate for $j(t)$. We note that in order to demonstrate that h has compact essential numerical support we need to find an estimate of $j(t)$ that is dependent on t, otherwise we will not be able to calculate the values of t for which $j(t)$ becomes insignificant. Consider equation (5.103), if we were to make estimates by taking the absolute value into the integral, we would lose the e^{itx} term and our expression would be independent of t; such an estimate would be too crude.

We perform estimates on $j(t)$:

$$
\begin{aligned}
|j(t)| &= \left|\int_{-c}^{c} \hat{g}(x)\cosh(\lambda x)\, e^{itx}\,\mathrm{d}x\right| \\
&= \left|\int_{-c}^{c} \hat{g}(x)\cosh(\lambda x)\left[\cos(tx) + i\sin(tx)\right]\,\mathrm{d}x\right| \\
&\leq \left|\int_{-c}^{c} \hat{g}(x)\cosh(\lambda x)\cos(tx)\,\mathrm{d}x\right| + \left|i\int_{-c}^{c} \hat{g}(x)\cosh(\lambda x)\sin(tx)\,\mathrm{d}x\right| \\
&\leq \int_{-c}^{c} \left|\hat{g}(x)\cosh(\lambda x)\cos(tx)\right|\,\mathrm{d}x + \int_{-c}^{c} \left|\hat{g}(x)\cosh(\lambda x)\sin(tx)\right|\,\mathrm{d}x \\
&\leq M\int_{-c}^{c} \cosh(\lambda x)\left|\cos(tx)\right|\,\mathrm{d}x + M\int_{-c}^{c} \cosh(\lambda x)\left|\sin(tx)\right|\,\mathrm{d}x,
\end{aligned}
\tag{5.104}
$$

where $M = \max_{x\in[-c,c]} |\hat{g}(x)|$. We know that $\hat{g}$ is bounded since $g \in L^1(\mathbb{R})$.

When considering possible candidates for $\hat{g}$ in the latter part of Chapter 4, we noted that $\hat{g}$ needed to have exponential decay (or be zero) at the far field. We noted also that for $\hat{g}$ with Gaussian decay outside some interval $[-a, a]$, if $\hat{g}$ is constant on $[-a, a]$ then $g \notin F_{[\lambda,p]}$. In the final step in equation (5.104) we are treating $\hat{g}$ as a constant on the interval $[-c, c]$. Equation (5.104) will not have compact essential numerical support (both integrals tend to a constant, dependent of the value of c, as $t \to \infty$). We require the properties of $\hat{g}$ inside the integral to ensure that $j(t)$ has compact essential numerical support. We can demonstrate that $j(t)$ has compact essential numerical support by making use of the Riemann-Lebesgue Lemma (see Theorem 4.12). We will also demonstrate in the next section that $j(t)$ has compact essential numerical support by means of an example.

Referring back to the equation for $j(t)$, we have:

$$j(t) = \int_{-c}^{c} \hat{g}(x) \cosh(\lambda x) e^{itx} \, \mathrm{d}x. \tag{5.105}$$

Now, $\hat{g}\xi_\lambda$ is defined as belonging to $L^p(\mathbb{R})$ and on the interval $[-c, c]$, $\hat{g}\xi_\lambda \in L^p[-c, c]$. It follows that $\hat{g}\xi_\lambda \in L^1[-c, c]$ and hence the Riemann-Lebesgue Lemma holds. We deduce that $|j(t)| \to 0$ as $|t| \to \infty$, that is, j has compact essential numerical support. We could use the method of proof of the Riemann-Lebesgue Lemma, p.103 in Rudin [71] to try and generate estimates for j. Rudins method approximates a function by trigonometric polynomials. We note, however, that the estimates will be independent of t.

Combining the above, we have demonstrated that $f(t)$ and $j(t)$ have compact essential numerical support. We deduce that the relaxation spectrum $h(t) = f(t) + j(t)$ which is defined as: $h = \mathscr{F}^{-1}[\hat{g}\cosh]$ has compact essential numerical support.

5.5 Accuracy of Estimation for an Example

In this section we use the calculations in the previous section to calculate an interval of compact essential numerical support for an example. The example we use is the relaxation spectrum h_2 that we calculated at the beginning of this chapter. We calculate an interval of compact essential numerical support and compare it with the actual relaxation spectrum to evaluate the accuracy of our method.

We refer the reader to the function h_2 that we calculated at the beginning of the chapter (equation (5.3)):

$$\begin{aligned} h_2(t) &= \int_{-\infty}^{-2} e^{-x^2}\left(\frac{e^{\lambda x}+e^{-\lambda x}}{2}\right) e^{ixt}\,\mathrm{d}x + \int_{-2}^{2} e^{-c\epsilon^2|x|}\left(\frac{e^{\lambda x}+e^{-\lambda x}}{2}\right) e^{ixt}\,\mathrm{d}x \\ &+ \int_{2}^{\infty} e^{-x^2}\left(\frac{e^{\lambda x}+e^{-\lambda x}}{2}\right) e^{ixt}\,\mathrm{d}x, \end{aligned} \tag{5.106}$$

for $c = 2$, $\epsilon = 1$ and $\lambda = \pi/2$ (we use these values to be able to compare directly with Figure 5.2).

We perform calculations for $f(t)$:

$$\begin{aligned} |f(t)| &\leq \frac{\sqrt{\pi}}{2\epsilon}\exp\left\{-c^2\epsilon^2\right\}\left\{e^{-c\lambda}\left|\cos(ct)\Re(w(z_4))\right| + e^{-c\lambda}\left|\sin(ct)\Im(w(z_4))\right|\right. \\ &+ \left. e^{c\lambda}\left|\cos(ct)\Re(w(z_2))\right| + e^{c\lambda}\left|\sin(ct)\Im(w(z_2))\right|\right\} \\ &\leq \frac{\sqrt{\pi}}{2\epsilon}\exp\{-4\}\left\{e^{-\pi}\left|\Re(w(z_4))\right| + e^{-\pi}\left|\Im(w(z_4))\right|\right. \\ &+ \left. e^{\pi}\left|\Re(w(z_2))\right| + e^{\pi}\left|\Im(w(z_2))\right|\right\} \\ &\leq \frac{\sqrt{\pi}}{2}\exp\{-4\}\left\{e^{-\pi}\left[\left|\Re(w(z_4))\right| + \left|\Im(w(z_4))\right|\right]\right. \\ &+ \left. e^{\pi}\left[\left|\Re(w(z_2))\right| + \left|\Im(w(z_2))\right|\right]\right\}, \end{aligned} \tag{5.107}$$

which has the following graph:

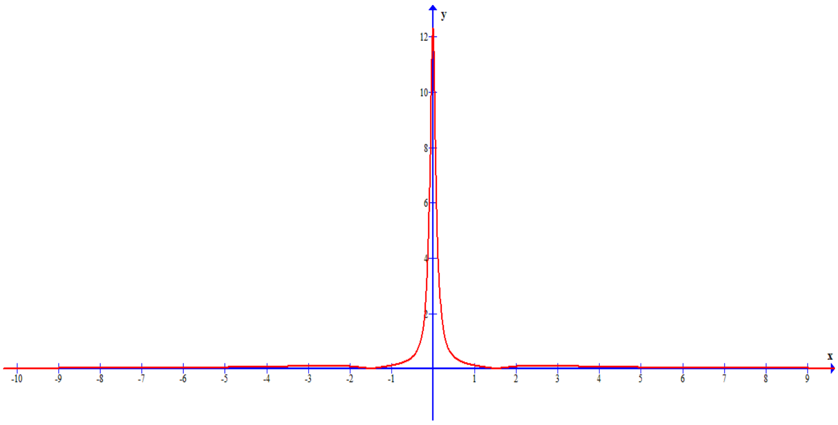

Figure 5.5: $f(x)$ for $c = 2$, $\epsilon = 1$ and $\lambda = \pi/2$

We can deduce that $|f(x)|$ becomes insignificant for x outside the interval $[-4, 4]$. Noting that $x = t/2\epsilon = t/2$, we deduce that $|f(t)|$ is insignificant for t outside the interval $[-8, 8]$. We interpret this as f being numerically supported on $[-8, 8]$. We note also that $|f(x)| \leq 1$ outside the interval $[-0.30275, 0.30275]$ and hence $|f(t)| \leq 1$ outside the interval $[-0.6055, 0.6055]$.

Since we were unable to obtain an estimate of the interval of compact essential numerical support for j, we will consider the contribution, C_2 that j makes to h that we calculated at the beginning of this chapter, see equation (5.23).

$$\begin{aligned} C_2 &= \frac{1}{2}\left[\frac{e^{(it+\beta)c}-1}{it+\beta} + \frac{e^{(it-\alpha)c}-1}{it-\alpha} + \frac{1-e^{-(it+\alpha)c}}{it+\alpha} + \frac{1-e^{-(it-\beta)c}}{it-\beta}\right] \\ &= \frac{(\alpha-\beta)\,t^2 + \alpha\beta\,(\beta-\alpha) + \left[t^3\left(e^{-c\alpha}-e^{c\beta}\right) + (\alpha-\beta)\,t\left(\alpha e^{c\beta}-\beta e^{-c\alpha}\right)\right]\sin(ct)}{t^4 + (\alpha^2+\beta^2)\,t^2 + \alpha^2\beta^2} \\ &+ \frac{\left[t^2\left(\beta e^{c\beta}-\alpha e^{-c\alpha}\right) + \alpha\beta\left(\alpha e^{c\beta}-\beta e^{-c\alpha}\right)\right]\cos(ct)}{t^4 + (\alpha^2+\beta^2)\,t^2 + \alpha^2\beta^2}, \end{aligned} \tag{5.108}$$

for $\alpha = \lambda + c\epsilon^2$ and $\beta = \lambda - c\epsilon^2$. C_2 has the following graph (green curve), where we have included the graphs for h_2 (blue curve) see Figure 5.2 and $f(t)$ (red curve):

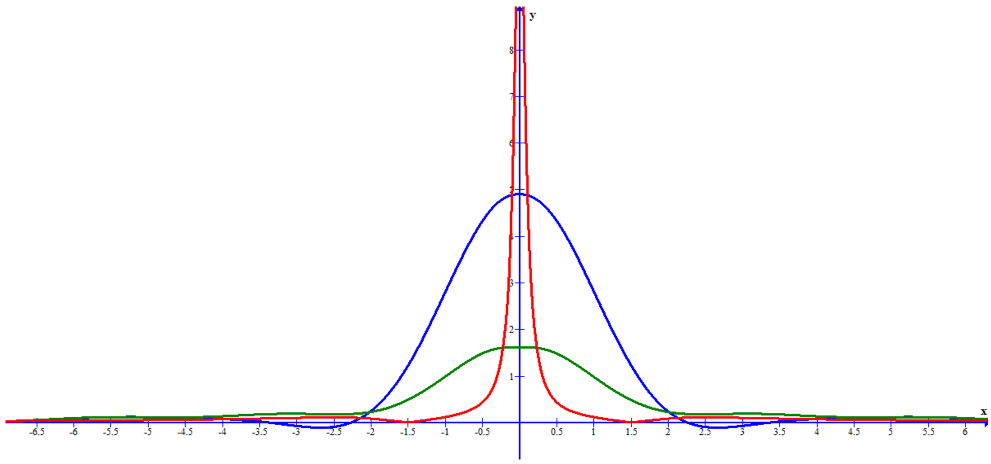

Figure 5.6: C_2, h_2 and f for $c = 2$, $\epsilon = 1$ and $\lambda = \pi/2$.

For this example we note that h_2 and C_2 appear to become insignificant outside the interval $[-4, 4]$. Hence, for this example, the contribution that j makes to h_2 is sufficient to estimate the true interval of compact essential numerical support for h_2. However, as we mentioned before, we might wish to calculate some L^p norm of h. In this example, the j estimate might be sufficient to estimate an interval of compact essential numerical support, but the integral of C_2 would drastically underestimate the integral of h_2. It would appear that we would need the f contribution in addition to the j contribution if we were to accurately estimate the integral of h_2. This may not be the case for all examples. More examples would need to be done to evaluate to what extent the j estimate is sufficient to estimate properties of h. We could also consider the contribution to h_2 from f (i.e. $C_1 + C_3$ as calculated in equation (5.21)) and evaluate what values c and ϵ need to be to ensure that f does not contribute very much to the relaxation spectrum h.

5.6 Summary

In this chapter we have calculated the corresponding relaxation spectra for the functions that we introduced in Chapter 4. We demonstrated that (visually) they all have compact essential numerical support. We did note however that the decay at far field varied for different types of functions.

By making a formal definition of what we mean by compact essential numerical support we were able to consider how one might tackle the problem of demonstrating and then calculating an interval of compact essential numerical support for the relaxation spectrum. It is not obvious how to obtain the best possible estimates of compact essential numerical support. Hence, we have considered the possible norms and methods that we might use to estimate an interval of compact essential numerical support. Since we are defining the concept of compact essential numerical support as an interval, outside which the relaxation spectrum becomes insignificant, it made sense to consider the supremum norm in our estimates. However, should one wish to obtain additional information about the rate of decay of the relaxation spectrum, we have suggested a method using a weighted integral norm.

We have demonstrated via calculations that the relaxation spectrum does have compact essential numerical support. Furthermore, we applied our method to an example, using one of the spectra (h_2) that we had calculated at the beginning of this chapter. We demonstrated that f, the far-field contribution to h_2 did not gives us any additional information about the compact essential numerical support of h_2 and that for this particular example, the j contribution to h_2 gave us all the information we required. However, more examples need to be considered to determine to what extent this is true for other possible functions. It would be interesting to calculate the contribution that f makes to h for different values of c and ϵ to determine when it might make a significant contribution to the relaxation spectrum. Clearly there are many ways in which one could further develop the idea of compact essential numerical support. There may be ways to make better estimates that give more accuracy and/or are less computationally demanding.

Chapter 6

Conclusions and Further Work

6.1 Summary

The purpose of this thesis was to try and recover the relaxation spectrum for a viscoelastic fluid and deduce certain properties about it. Part of the research was to put previous results in this area into context. The work of the authors Davies, Anderssen, Loy, Newbury,[50],[51], [20], Dodd [23] and Renardy [69], has been evaluated in detail. We have filled in gaps in their work and considered the extent of the validity of some of their results. We saw from the beginning that there were discrepancies in the work of Davies, Anderssen, Loy, Newbury [50]. Their original result of compact support for the relaxation spectrum h, relied on the assumption that g was compactly supported. Renardy points out in his work [69] that for their problem as it is defined, g must be an analytic function and that the only analytic function with compact support is the zero function. Hence, the results of Davies, Anderssen, Loy, Newbury [50] and Dodd [23] are valid for the zero function only. This prompted Loy, Davies and Anderssen [51] to write a revised paper where g is no longer assumed to be compactly supported. We have considered this paper in great detail and it became clear that their revised result of compact support for the relaxation spectrum h should, at most, be considered as support in a weak sense.

We also made some remarks, briefly, with regards to the sampling localisation results of Davies and Anderssen [20]. We demonstrated that there were steps in their calculations that need further justification. We will consider this further in the next section.

In the original work by Davies, Anderssen, Loy and Newbury [50], in addition to the direct method they use for obtaining an interval of support for the relaxation spectrum h, they also perform calculations using the Paley-Wiener theorem. With the assumption that g is compactly supported, this calculation is quite straightforward. The interval of support they obtain for h, via the Paley-Wiener theorem, matches the interval that they obtained via their direct method. In the revised paper by Loy, Davies and Anderssen [51], the interval of support they obtain for h is in agreement with that of their original work. However, they make no reference to the Paley-Wiener theorem.

We attempted to use the Paley-Wiener theorem to obtain a similar interval of compact support for h, where g was no longer compactly supported. Despite modifications and mollifications it proved difficult to satisfy the Paley-Wiener theorem when g was not compactly supported. Eventually, by assuming that the criteria of the Paley-Wiener theorem could be satisfied, we were able to prove some interesting results. We have proven that for $g \in F_{[\lambda,2]}$, with $\hat{g} \cdot \xi_\lambda$ being entire and of exponential type, the only possible solution is $g = 0$. This is an interesting result. However, it does not directly tell us that the relaxation spectrum does not have compact support (as the Paley-Wiener theorem is a sufficient, not necessary condition).

By considering a larger space of functions, the space of tempered distributions, we are able to prove, using the Paley-Wiener-Schwartz theorem, a stronger result than that of the Paley-Wiener theorem, that the relaxation spectrum cannot have compact support for the spaces of functions we are working in. Since L^p is a subspace of the space of tempered distributions $\mathscr{S}'$, we can deduce that the relaxation spectrum cannot have compact support for the L^p/L^q setting. This is the setting for the results of Loy, Davies and Anderssen and Newbury [51], [50].

Numerical evidence suggests the sampling localisation results of Davies and Anderssen are still useful. In Chapter 5 we examined to what extent h has compact support "to machine tolerance", that is, the relaxation spectra become insignificant (visually) outside some closed interval. We call this compact essential numerical support. We demonstrated for a class of examples, that the relaxation spectrum does in fact have compact essential numerical support. We have concentrated on results using the supremum norm which seems the natural norm to use for our concept of compact essential numerical support. Our results are practical, since computations involving the relaxation spectrum would ignore values once they are below some threshold value, which would be interpreted as the relaxation spectrum having compact support. The work we have done is only a first step towards a result for practical rheologists; it remains to investigate different norms, choosing weight functions appropriate to certain decay rates.

6.2 Further Work and Future Directions

In this section we review some of the work that we have mentioned in the thesis and we also consider some outstanding problems.

6.2.1 Sampling Localisation

At the end of Chapter 2 we evaluated the sampling localisation results of Davies and Anderssen [20]. The reason for considering these results was in response to a paper by Renardy [69] where he brings to light an important fact that this interval that Davies and Anderssen believe to be the smallest possible, cannot be so. The function g is an analytic function, that is, it can be determined everywhere by its values in any interval. We demonstrated in Chapter 2 that one of the steps in their calculations where they take a limit into an integral may not be justified, since we have demonstrated that the Dominated Convergence Theorem cannot be satisfied.

We refer the reader to equations (2.75) to (2.78) in section 2.5.2 of Chapter 2. Assuming for the moment that $\lim_{\epsilon\to 0} \Psi_{\epsilon}^{'}(s) = \Psi^{'}(s)$, then, Davies and Anderssen proceed in their calculations by considering the expressions that they obtain for both the storage and loss moduli, seen in equations (2.65) and (2.66). They note that both are expressions of the form:

$$\operatorname{erf}\left(\frac{x + (\pi i/2)}{\sqrt{2}\epsilon}\right) \tag{6.1}$$

We noted at the end of Chapter 2 that we would consider their results in greater detail in this chapter. In Chapter 2 we questioned the validity of taking the limit as ϵ tends to zero of $\Psi_{\epsilon}^{'}(s)$. In the work that follows, Davies and Anderssen make further calculations involving limits as ϵ tends to zero of expressions involving equation (6.1). By considering the properties of equation (6.1), we demonstrate that the final steps in Davies and

Anderssens calculations, in obtaining Theorems 1.1 and 1.2, should really be considered as results in the sense of distributions; pointwise limits are not really appropriate. Davies and Anderssen define the elementary sampling function $E^*_\epsilon(x)$ as:

$$E^*_\epsilon(x) = E^{'}_\epsilon(x) + iE^{''}_\epsilon(x) \equiv \frac{1}{\pi}\mathrm{erf}\left(\frac{x + (\pi i/2)}{\sqrt{2}\epsilon}\right),\ \epsilon > 0. \tag{6.2}$$

They differentiate $E^*_\epsilon(x)$ with respect to x, to obtain:

$$\begin{aligned} DE^*_\epsilon = \frac{\mathrm{d}E^*_\epsilon}{\mathrm{d}x} &= \sqrt{\frac{2}{\pi}}\frac{1}{\epsilon}\exp\left\{-\frac{(x + (\pi i/2))^2}{2\epsilon^2}\right\} \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{\epsilon}\exp\left\{-\frac{(x^2 - (\pi^2/4))}{2\epsilon^2}\right\}\exp\left\{\frac{\pi x i}{2\epsilon^2}\right\}. \end{aligned} \tag{6.3}$$

Davies and Anderssen note that as ϵ tends to zero, the derivative tends to zero (exponentially) for $|x| > \pi/2$. From this they claim that $E^*_0(x)$ is constant for $|x| > \pi/2$.

However, we must note that both $E^*_0(x)$ and DE^*_0 need to be interpreted as distributions, with the latter a distributional derivative. E^*_ϵ is a function of rapid oscillations near the origin for small ϵ; a sensible notation of limit would be a distribution or measure. In evaluating the derivative above, Davies and Anderssen are assuming that:

$$\lim_{\epsilon\to 0} DE^*_\epsilon = DE^*_0. \tag{6.4}$$

It may be true that $\lim_{\epsilon\to 0} E^*_\epsilon = E^*_0$. However, it is not obvious that equation (6.4) can be justified.

For a function $\varphi \in \mathscr{D}$ (the space of compactly supported test functions) supported on $[a, b]$, we need to satisfy the following (we note that this is a formal calculation):

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_a^b DE_\epsilon^* (x) \varphi (x) &= \lim_{\epsilon \to 0} \int_a^b \frac{\mathrm{d}E_\epsilon^* (x)}{\mathrm{d}x} \varphi (x) \, \mathrm{d}x \\
&= - \lim_{\epsilon \to 0} \int_a^b E_\epsilon^* (x) \frac{\mathrm{d}\varphi (x)}{\mathrm{d}x} \, \mathrm{d}x \\
&= - \int_a^b \lim_{\epsilon \to 0} E_\epsilon^* (x) \frac{\mathrm{d}\varphi (x)}{\mathrm{d}x} \, \mathrm{d}x \\
&= - \int_a^b E_0^* (x) \frac{\mathrm{d}\varphi (x)}{\mathrm{d}x} \, \mathrm{d}x \\
&= \int_a^b \frac{\mathrm{d}E_0^* (x)}{\mathrm{d}x} \varphi (x) \, \mathrm{d}x \\
&= \int_a^b DE_0^* (x) \varphi (x) \, \mathrm{d}x \qquad (6.5)
\end{aligned}$$

The problem with equation (6.5) is that we cannot justify taking the limit inside the integral unless we can satisfy the Dominated Convergence theorem. To satisfy the Dominated Convergence theorem, we need to demonstrate that:

$$\left| \frac{\mathrm{d}E_\epsilon^* (x)}{\mathrm{d}x} \varphi (x) \right| = \left| E_\epsilon^* (x) \frac{\mathrm{d}\varphi (x)}{\mathrm{d}x} \right| \leq \Phi (x) \qquad (6.6)$$

where $\Phi (x)$ is integrable and independent of ϵ.

Consider the following:

$$\begin{aligned}
\left|\frac{\mathrm{d}E_{\epsilon}^{*}\left(x\right)}{\mathrm{d}x}\varphi\left(x\right)\right| &= \left|\sqrt{\frac{2}{\pi}}\frac{1}{\epsilon}\exp\left\{-\frac{\left(x^{2}-\left(\pi^{2}/4\right)\right)}{2\epsilon^{2}}\right\}\exp\left\{\frac{\pi x i}{2\epsilon^{2}}\right\}\varphi\left(x\right)\right| \\
&= \sqrt{\frac{2}{\pi}}\frac{1}{\epsilon}\left|\exp\left\{-\frac{\left(x^{2}-\left(\pi^{2}/4\right)\right)}{2\epsilon^{2}}\right\}\varphi\left(x\right)\right| \qquad (6.7)
\end{aligned}$$

The exponential term in equation (6.7) is bounded for $|x| \geq \pi/2$. However, for values of x such that $|x| < \pi/2$ the expression above increases exponentially as $\epsilon \to 0$. This would suggest that we cannot find a dominator and hence, it may not be possible to take the limit inside the integral.

Finally, assuming that the steps we have referred to above can in fact be justified, referring back to equations (2.45) and (2.65) and noting that $\phi^{'}\left(\omega\right) = \omega^{2}\varphi^{'}\left(\omega\right)$, Davies and Anderssen consider

$$\begin{aligned}
\eta_{ab} &= \int_{0}^{\infty} \omega^{-2}\varphi'\left(\omega\right)G^{'}\left(\omega\right)\,\mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} \lim_{\epsilon\to 0}\left[E_{\epsilon}^{'}\left(\ln\left(b\omega\right)\right) - E_{\epsilon}^{'}\left(\ln\left(a\omega\right)\right)\right]\omega^{-1}G^{'}\left(\omega\right)\,\mathrm{d}\ln\omega. \qquad (6.8)
\end{aligned}$$

Davies and Anderssen attempt to obtain information about the interval required for ω to recover the relaxation spectrum on an interval $[a, b]$ by considering the limit as ϵ tends to zero in the above expression. What they deduce is that:

$$\lim_{\epsilon\to 0} E_{\epsilon}^{'}\left(\ln\left(b\omega\right)\right) - E_{\epsilon}^{'}\left(\ln\left(a\omega\right)\right) = 0 \qquad (6.9)$$

for values of ω in the interval:

$$-\frac{\pi}{2} - \ln b < \ln \omega < \frac{\pi}{2} - \ln a. \tag{6.10}$$

They use results (7.1.16, Abramowitz and Stegun [2], page 298 and Olde Daalhuis et al. [64]) concerning the limit of the error function of a complex variable, based on asymptotics.

We conclude that their results [20], despite being interesting and offering a means to tackle the problem of recovering information about the relaxation spectrum, may be misleading. There is definitely a need to clarify to what extent their results are valid. It might be that these results are useful and hence trying to justify some of the steps in their calculations could be beneficial.

6.2.2 The Connection with Wavelets Methods

In Chapter 2, we established that $\langle \kappa_g(\epsilon), f \rangle = \int_{-\infty}^{\infty} W_\epsilon (f * g)$, and were able to show that

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} W_\epsilon (f * g) = \lim_{\epsilon \to 0} \int_{|s| \le \lambda + \delta} W_\epsilon (f * g). \tag{6.11}$$

However, we did not attempt to evaluate the limit as ϵ tends to zero of W_ϵ; the ϵ divisor makes this a challenging calculation. There is a sense in which W_ϵ can be viewed as a wavelet scaling function (see Davies and Goulding [22]). The regularisation, then limiting process, has been studied by Davies and Goulding [22]. Assuming the relaxation spectrum h is square integrable, it can be decomposed into an integral over all scales of its wavelet transforms. One can truncate the domain of the integral, using a Calderon-Mallet decomposition; roughly speaking, the integral Calderon term can be thought of as describing the graph of h, whilst the Mallet remainder term gives the integral of h. One can approximate the relaxation spectrum. There are some very interesting mathematical questions here that have only in part been answered.

6.2.3 Fourier Multipliers

In this section we make some remarks with regards to Fourier multipliers.

We define (taken from Weis [82]) what we mean a Fourier multiplier.

Definition 6.1. *Let X and Y be Banach spaces, $B(X,Y)$ be the space of bounded linear operators from X to Y, and $\mathscr{S}(X)$ be the space of rapidly decreasing functions from $\mathbb{R}$ to X. For $f \in L^1(\mathbb{R}, X)$, we write:*

$$\mathscr{F}[f(t)] = \int f(s)\, e^{-its}\, \mathrm{d}s \tag{6.12}$$

for the Fourier transform of f and $\mathscr{F}^{-1}[f]$ for the inverse Fourier transform of f. We say that a function $M : \mathbb{R} \to B(X,Y)$ is a Fourier multiplier on $L^p(\mathbb{R}, X)$ if the expressions

$$Kf = \mathscr{F}^{-1}\left[M\left(\cdot\right)\mathscr{F}\left[f\left(\cdot\right)\right]\right] \tag{6.13}$$

where $f \in \mathscr{S}\left(X\right)$ *are well defined and* K *extends to a bounded operator* $K : L^p\left(\mathbb{R}, X\right) \to L^p\left(\mathbb{R}, Y\right)$.

Referring back to our problem; for $Kg = \mathscr{F}^{-1}\left[\xi_\lambda \cdot \hat{g}\right]$, it may be shown that ξ_λ is a Fourier multiplier in the L^2 setting (the largest multiplier space in the L^p/L^q setting) if and only if ξ_λ is bounded and measurable. Clearly, cosh is not bounded.

Although our multiplier $\cosh\left(\lambda x\right)$ does not satisfy the criteria for a Fourier multiplier in a classical sense, it may be possible to extend the classical theory.
For more details on Fourier Multipliers see Weis [82], McConnell [55] and Triebel [80].

6.2.4 The Space of Exponentially Decaying Test Functions

In Chapter 3 and Chapter 4 we have proven, using Plancherel and Paley-Wiener type theorems, that there are no non-trivial relaxation spectra with compact support. We have been working with functions in the space:

$$F_{\left[\lambda, \mathscr{D}'\right]} = \left\{g \in L^1\left(\mathbb{R}\right) \mid \xi_\lambda \hat{g} \in \mathscr{D}'\left(\mathbb{R}\right)\right\}, \tag{6.14}$$

where $\mathscr{D}'\left(\mathbb{R}\right)$ denotes the bounded linear functionals on the test space $\mathscr{D}\left(\mathbb{R}\right)$.

For $\mathscr{D}'(\mathbb{R}) = L^2(\mathbb{R})$, we were able to prove, using the Plancherel theorem and the Paley-Wiener theorem, that if $|\hat{g} \cdot \xi_\lambda|$ is of exponential type, then $|\hat{g}|$ is of exponential type. Furthermore, it follows from the Paley-Wiener theorem that g must be zero and hence h is zero. For $\mathscr{D}'(\mathbb{R}) = \mathscr{S}'(\mathbb{R})$, we were able to prove, using properties of tempered distributions (see Theorem 8.3.2 in Friedlander and Joshi [29]) and the Paley-Wiener-Schwartz theorem, that if h has compact support then $|\hat{g} \cdot \xi_\lambda|$ is of exponential type, from which we prove that $|\hat{g}|$ is of exponential type. Furthermore, it follows from the Paley-Wiener-Schwartz theorem that g must be zero and hence h is zero. We also deduce that since $L^p(\mathbb{R})$ is a subspace of $\mathscr{S}'(\mathbb{R})$, that the above results hold for the L^p/L^q setting.

One might ask: are there any spaces of functions $\mathscr{D}'(\mathbb{R})$ for which we can obtain results of compact support for the relaxation spectrum? One possibility would be to consider the space of distributions on the space of exponentially decaying, or Gaussian decaying, test functions.

Plancherel and Paley-Wiener type theorems would need to be developed for this space of distributions in order to perform similar calculations as those we mentioned above, to derive an interval of support for h. If we consider the results of Chapter 3 and Chapter 4; to obtain a non-trivial relaxation spectrum h with compact support, we require a space of functions $F_{[\lambda,\mathscr{D}']}$ such that g compactly supported and non-trivial is not an empty space. Assuming that we have such a g, i.e. $g \in L^1(\mathbb{R})$ and has compact support, then we can deduce from results in Chapter 3 that $\hat{g}$ cannot have exponential decay. Clearly, $\hat{g} \cdot \xi_\lambda \notin L^p(\mathbb{R})$ and $\hat{g} \cdot \xi_\lambda \notin \mathscr{S}'(\mathbb{R})$. However it is plausible that $\hat{g}\xi_\lambda \in \mathscr{D}'(\mathbb{R})$, the space of distributions on test functions of exponential (or Gaussian) decay. Moreover one might be able to demonstrate a Plancherel-type theorem, that is $\mathscr{F} : \mathscr{D}'(\mathbb{R}) \to \mathscr{D}'(\mathbb{R})$ is a well-behaved linear mapping, analogous to results that are true for $\mathscr{S}'(\mathbb{R})$.

Another space of functions that one could consider, is the space of distributions on the compactly supported, continuous test functions $C_0(\mathbb{R})$. The space of distributions on these test functions, $(C_0(\mathbb{R}))'$ must then correspond to the space of finite Radon measures by the Riesz-Markov Theorem. Hence, we would need to impose that:

$$\mu(B) = \int_B \hat{g} \cdot \xi_\lambda \tag{6.15}$$

is a finite Radon measure to be able to consider results of non-trivial h with compact support. A similar assumption was made in Proposition 4.11, giving sufficient conditions for the relaxation spectrum to be strictly positive definite. It may be that the assumption that $g \in L^1(\mathbb{R})$ needs to be relaxed to develop an appropriate theory.

6.2.5 Compact Essential Numerical Support

In Chapter 5 we introduced the concept of "compact essential numerical support". There are only a few remarks regarding this idea in the literature, one of these being in the work of Grip and Pfander [35]. We have demonstrated, using examples and direct calculations on classes of examples, that the relaxation spectrum has compact essential numerical support. However, these calculations would be quite difficult for an engineer who may simply want a quick way to calculate an interval of numerical support. It is clear that there is room for improvement here with regards to these estimates. Ideally, one would like to develop other approaches. It would also be worthwhile to consider a few different norms to see which ones yield most information about the problem, but also which ones are less computationally demanding.

Chapter 7

Appendix

7.1 Fourier Transform 1

$$\mathscr{F}\left[\operatorname{sech} y\right](p) \;=\; \int_{-\infty}^{\infty} \exp\left(-ipy\right)\,\operatorname{sech} y\;\mathrm{d}y \;=\; \int_{-\infty}^{\infty} \frac{2e^{-ipy}}{e^{y}+e^{-y}}\,\mathrm{d}y \tag{7.1}$$

We solve using contour integration methods and residues.

Note that;

$$e^{y} = e^{-i^2 y} = e^{-i(iy)} = \cos iy - i\sin iy. \tag{7.2}$$

Similarly

$$e^{-y} = e^{i^2 y} = e^{i(iy)} = \cos iy + i\sin iy, \tag{7.3}$$

hence

$$e^{-y} + e^{y} = 2\cos iy. \tag{7.4}$$

We have;

$$\mathscr{F}\left[\operatorname{sech} y\right] \; = \; \int_{-\infty}^{\infty} \frac{2e^{-ipy}}{2\cos iy}\,\mathrm{d}y = \int_{-\infty}^{\infty} \frac{e^{-ipy}}{\cos iy}\,\mathrm{d}y. \tag{7.5}$$

Now, if we consider the problem;

$$\int_C \frac{e^{-pz}}{\cos z}\,\mathrm{d}z, \tag{7.6}$$

where p is a constant. C is the contour given in Figure 7.1.

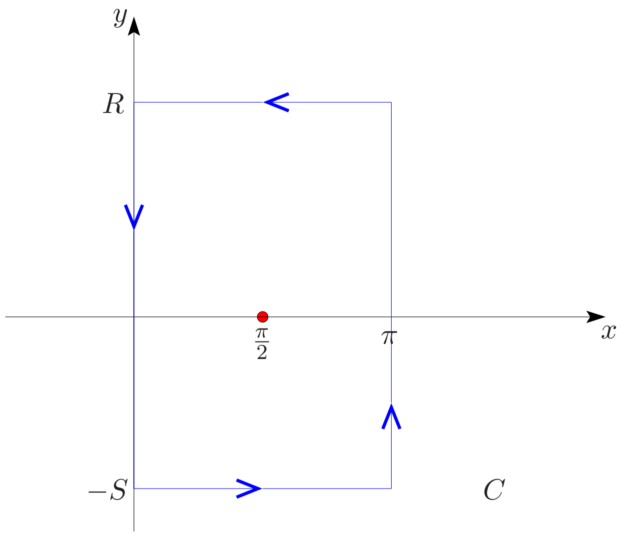

Figure 7.1: The contour, C

The value of the integral over the contour C is equal to zero if there are no points of singularity within or on the track of the contour C. If singularities occur then the integral is equal to the sum of the residues of the poles multiplied by $2i\pi$.

Singularities occur when $\cos z = 0 \Rightarrow$ we have singularities at

$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

That is, we have multiple poles. However, only $z = +\frac{\pi}{2}$ is in the region contained within the contour C.

Now

$$\int_C \frac{e^{-pz}}{\cos z}\,\mathrm{d}z = 2\pi i \text{residue}\left\{f(z); z = \frac{\pi}{2}\right\}. \tag{7.7}$$

The following theorem was taken from Priestley [68]:

Theorem 7.1. *Residue Theorem for Covert Simple Pole*

$$If\ \ f(z) = \frac{h(z)}{k(z)}, h(a) \neq 0, k(a) = 0, then\ res\,\{f(z); a\} = \frac{h(a)}{k'(a)},$$

this implies that

$$\text{res}\left\{f(z); \frac{\pi}{2}\right\} = \frac{e^{\frac{-a\pi}{2}}}{-\sin(\frac{\pi}{2})} = -e^{\frac{-a\pi}{2}}.$$

We have;

$$\int_C \frac{e^{-pz}}{\cos z}\,\mathrm{d}z = 2i\pi\left(-e^{\frac{-p\pi}{2}}\right). \tag{7.8}$$

We proceed in calculating the LHS of the above expression in the remainder of this problem which will allow us to evaluate the expression (7.5) that we require. That is, if we integrate around the contour we obtain:

$$\begin{aligned}\int_C \frac{e^{-pz}}{\cos z}\,\mathrm{d}z = \int_0^{\pi} \frac{e^{-p(x-iS)}}{\cos(x-iS)}\,\mathrm{d}x + \int_{-S}^{R} \frac{e^{-p(\pi+iy)}}{\cos(\pi+iy)} i\,\mathrm{d}y \\ + \int_{\pi}^{0} \frac{e^{-p(x+iR)}}{\cos(x+iR)}\,\mathrm{d}x + \int_{R}^{-S} \frac{e^{-p(0+iy)}}{\cos(0+iy)} i\,\mathrm{d}y. \end{aligned} \tag{7.9}$$

We consider the two integrals;

$$\int_0^{\pi} \frac{e^{-p(x-iS)}}{\cos(x-iS)}\,\mathrm{d}x, \tag{7.10}$$

and

$$\int_{\pi}^{0} \frac{e^{-p(x+iR)}}{\cos(x+iR)}\,\mathrm{d}x. \tag{7.11}$$

Considering (7.10)

$$\begin{aligned}\left|\int_0^{\pi} \frac{e^{-p(x-iS)}}{\cos(x-iS)}\,\mathrm{d}x\right| &= \int_0^{\pi} \left|\frac{e^{-p(x-iS)}}{\cos(x-iS)}\right| \mathrm{d}x = \int_0^{\pi} \left|\frac{2e^{-p(x-iS)}}{e^{i(x-iS)} + e^{-i(x-iS)}}\right| \mathrm{d}x \\ &\leq \int_0^{\pi} \frac{2\left|e^{-px}\right|\left|e^{ipS}\right|}{\left|e^{i(x-iS)}\right| - \left|e^{-i(x-iS)}\right|}\,\mathrm{d}x \\ &\leq \int_0^{\pi} \frac{2e^{|p|\pi}}{e^{S} - e^{-S}}\,\mathrm{d}x, \end{aligned} \tag{7.12}$$

we note that

$$\int_0^{\pi} \frac{2e^{|p|\pi}}{e^S - e^{-S}}\,\mathrm{d}x \to 0$$

as $S \to \infty$.

Similarly, considering (7.11), we have

$$\begin{aligned}
\left|\int_{\pi}^{0} \frac{e^{-p(x+iR)}}{\cos(x+iR)}\,\mathrm{d}x\right| &\leq \left|\int_{\pi}^{0} \left|\frac{e^{-p(x+iR)}}{\cos(x+iR)}\right| \mathrm{d}x\right| = \left|\int_{\pi}^{0} \left|\frac{2e^{-p(x+iR)}}{e^{i(x+iR)} + e^{-i(x+iR)}}\right| \mathrm{d}x\right| \\
&\leq \left|\int_{\pi}^{0} \frac{2\left|e^{-px}\right|\left|e^{-ipR}\right|}{\left|\left|e^{i(x+iR)}\right| - \left|e^{-i(x+iR)}\right|\right|}\,\mathrm{d}x\right| \\
&\leq \left|\int_{\pi}^{0} \frac{2e^{|p|\pi}}{e^R - e^{-R}}\,\mathrm{d}x\right|. \qquad (7.13)
\end{aligned}$$

Note that

$$\left|\int_{\pi}^{0} \frac{2e^{|p|\pi}}{e^R - e^{-R}}\,\mathrm{d}x\right| \to 0,$$

as $R \to \infty$.

Now, if we take limits as $R \to \infty$ and $S \to \infty$, then, from (7.9) we get

$$\int_C \frac{e^{-pz}}{\cos z}\,\mathrm{d}z = \int_{-\infty}^{\infty} \frac{e^{-p(\pi+iy)}}{\cos(\pi+iy)} i\,\mathrm{d}y + \int_{\infty}^{-\infty} \frac{e^{-piy}}{\cos(iy)} i\,\mathrm{d}y. \qquad (7.14)$$

If we switch the limits of integration we get;

$$\int_C \frac{e^{-pz}}{\cos z}\,\mathrm{d}z = \int_{-\infty}^{\infty} \left(\frac{e^{-p(\pi+iy)}}{\cos(\pi+iy)} - \frac{e^{-piy}}{\cos(iy)}\right) i\,\mathrm{d}y. \qquad (7.15)$$

Noting that

$$\cos(\pi + iy) = \cos\pi\cos iy - \sin\pi\sin iy = -1\cos iy - 0 = -\cos iy,$$

then substituting this into equation (7.15) we obtain;

$$\begin{aligned}\int_C \frac{e^{-pz}}{\cos z}\,\mathrm{d}z &= \int_{-\infty}^{\infty}\left(\frac{e^{-p(\pi+iy)}}{-\cos(iy)} + \frac{e^{-piy}}{-\cos(iy)}\right) i\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\frac{(e^{-p\pi}+1)e^{-piy}}{-\cos(iy)} i\,\mathrm{d}y. \end{aligned} \tag{7.16}$$

Now, referring back to equation (7.8), we have that;

$$\int_C \frac{e^{-pz}}{\cos z}\,\mathrm{d}z = \int_{-\infty}^{\infty}\frac{(e^{-p\pi}+1)\,e^{-piy}}{-\cos(iy)} i\,\mathrm{d}y = 2i\pi\left(-e^{\frac{-p\pi}{2}}\right), \tag{7.17}$$

that is;

$$-\left(e^{-p\pi}+1\right)\int_{-\infty}^{\infty}\frac{e^{-piy}}{\cos(iy)} i\,\mathrm{d}y = 2i\pi\left(-e^{\frac{-p\pi}{2}}\right). \tag{7.18}$$

And after some rearrangement, we obtain;

$$\int_{-\infty}^{\infty}\frac{e^{-piy}}{\cos(iy)}\,\mathrm{d}y = \frac{2\pi\left(e^{\frac{-p\pi}{2}}\right)}{(e^{-p\pi}+1)} = \frac{2\pi}{e^{-\frac{p\pi}{2}} + e^{\frac{p\pi}{2}}}, \tag{7.19}$$

and

$$\frac{2\pi}{e^{\frac{p\pi}{2}}+e^{-\frac{p\pi}{2}}}=\pi\mathrm{sech}\frac{p\pi}{2}. \tag{7.20}$$

Note that the Fourier Transform in (7.5) is equal to the LHS of equation (7.19). Hence, we can write;

$$\mathscr{F}\left[\mathrm{sech}y\right]=\pi\mathrm{sech}\,\frac{p\pi}{2}, \tag{7.21}$$

as required.

7.2 Fourier Transform 2

$$\begin{aligned}\mathscr{F}\left[\frac{1}{\epsilon\sqrt{2\pi}}\exp\left(\frac{-x^2}{2\epsilon^2}\right)\right](p) &= \int_{-\infty}^{\infty}\frac{1}{\epsilon\sqrt{2\pi}}\exp\left(\frac{-x^2}{2\epsilon^2}\right)\exp\left(-ipx\right)\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\frac{1}{\epsilon\sqrt{2\pi}}\exp\left\{\frac{-x^2}{2\epsilon^2}-ipx\right\}\mathrm{d}x \end{aligned}\tag{7.22}$$

The probability density function (pdf) of a Normal distribution with mean μ and standard deviation σ , written N(μ,σ), takes the form;

$$f\left(x\right)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{\frac{-1}{2}\frac{\left(x-\mu\right)^2}{\sigma^2}\right\}. \tag{7.23}$$

From the properties of probability distributions we know that;

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1. \tag{7.24}$$

We use this result of the Normal distribution to solve the above Fourier transform problem. Consider equation (7.22), we can rewrite this as;

$$\begin{aligned}
\frac{1}{\epsilon\sqrt{2\pi}}\exp\left\{\frac{-1}{2}\left(\frac{x^2}{\epsilon^2}+2ipx\right)\right\} &= \frac{1}{\epsilon\sqrt{2\pi}}\exp\left\{\frac{-1}{2}\left(\frac{x^2+2ip\epsilon^2x+p^2\epsilon^4-p^2\epsilon^4}{\epsilon^2}\right)\right\}\\
&= \frac{1}{\epsilon\sqrt{2\pi}}\exp\left\{\frac{-1}{2}\left(\frac{x^2+2ip\epsilon^2x-p^2\epsilon^4}{\epsilon^2}\right)\right\}\exp\left\{\frac{-p^2\epsilon^4}{2\epsilon^2}\right\}\\
&= \frac{1}{\epsilon\sqrt{2\pi}}\exp\left\{\frac{-1}{2}\left(\frac{x+ip\epsilon^2}{\epsilon^2}\right)^2\right\}\exp\left\{\frac{-p^2\epsilon^4}{2\epsilon^2}\right\}, \qquad (7.25)
\end{aligned}$$

hence,

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{1}{\epsilon\sqrt{2\pi}}\exp\left\{\frac{-x^2}{2\epsilon^2}-ipx\right\}\mathrm{d}x &= \int_{-\infty}^{\infty}\frac{1}{\epsilon\sqrt{2\pi}}\exp\left\{\frac{-1}{2}\left(\frac{x+ip\epsilon^2}{\epsilon^2}\right)^2\right\}\exp\left\{\frac{-p^2\epsilon^4}{2\epsilon^2}\right\}\mathrm{d}x\\
&= \exp\left\{\frac{-p^2\epsilon^4}{2\epsilon^2}\right\}\int_{-\infty}^{\infty}\frac{1}{\epsilon\sqrt{2\pi}}\exp\left\{\frac{-1}{2}\left(\frac{x+ip\epsilon^2}{\epsilon^2}\right)^2\right\}\mathrm{d}x.
\end{aligned}$$

We compare the above equation with equation (7.23), and make use of the fact that the pdf of a normal distribution integrates to 1. Combining the above, we have that:

$$\mathscr{F}\left[\frac{1}{\epsilon\sqrt{2\pi}}\exp\left(\frac{-x^2}{2\epsilon^2}\right)\right] = \exp\left\{\frac{-p^2\epsilon^2}{2}\right\},$$

as required.

7.3 Fourier Transform 3

$$\begin{aligned}\mathscr{F}^{-1}\left[\operatorname{sech}(\alpha y)\cosh(\lambda y)\right] &= \int_{-\infty}^{\infty}\operatorname{sech}(\alpha y)\cosh(\lambda y)\,e^{ipy}\,\mathrm{d}y = \int_{-\infty}^{\infty}\frac{\cosh(\lambda y)}{\cosh(\alpha y)}e^{ipy}\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\frac{\cos(i\lambda y)}{\cos(i\alpha y)}e^{ipy}\,\mathrm{d}y \qquad (7.26)\end{aligned}$$

where $\alpha > \lambda$. We solve using contour integration methods and residues.

Consider the problem;

$$\int_{\Gamma}\frac{\cos(\lambda z)}{\cos(\alpha z)}e^{pz}\,\mathrm{d}z, \qquad (7.27)$$

where we treat p as a constant in the above integral. Γ is the contour given in Figure 7.3.

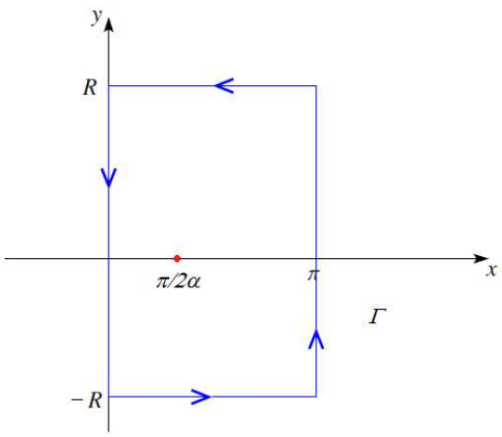

Figure 7.2: The contour, Γ

The value of the integral over the contour Γ is equal to zero if there are no points of singularity within or on the track of the contour Γ. If singularities occur then the integral is equal to the sum of the residues of the poles multiplied by $2i\pi$.

Singularities occur when $\cos(\alpha z) = 0$ which implies that we have singularities at

$z = \pm\frac{\pi}{2\alpha}, \pm\frac{3\pi}{2\alpha}, \pm\frac{5\pi}{2\alpha}, \ldots$

That is, we have multiple poles. However, only $z = \frac{\pi}{2\alpha}$ is in the region contained within the contour Γ.

Now

$$\int_\Gamma \frac{\cos(\lambda z)}{\cos(\alpha z)} e^{pz}\,\mathrm{d}z = 2\pi i \text{residue}\left\{f(z); z = \frac{\pi}{2\alpha}\right\}. \tag{7.28}$$

We refer the reader to the Residue Theorem for Covert Simple Pole (Theorem 7.1) that we introduced earlier.

Making use of this theorem, we can calculate:

$$\text{res}\left\{f(z); \frac{\pi}{2\alpha}\right\} = -\frac{e^{\pi p/2\alpha}\cos(\pi\lambda/2\alpha)}{\alpha\sin(\pi\alpha/2\alpha)} = -\frac{e^{\pi p/2\alpha}}{\alpha}\cos\left(\frac{\pi\lambda}{2\alpha}\right).$$

We have;

$$\int_\Gamma \frac{\cos(\lambda z)}{\cos(\alpha z)} e^{pz}\,\mathrm{d}z = -\frac{2\pi i}{\alpha} e^{\pi p/2\alpha}\cos\left(\frac{\pi\lambda}{2\alpha}\right). \tag{7.29}$$

We proceed in calculating the LHS of the above expression in the remainder of this problem which will allow us to evaluate the equation (7.26) that we require. That is, if we integrate around the contour we obtain:

$$\int_{\Gamma} \frac{\cos(\lambda z)}{\cos(\alpha z)} e^{pz}\,\mathrm{d}z = \int_0^{\pi} \frac{\cos(\lambda x - i\lambda R)}{\cos(\alpha x - i\alpha R)} e^{p(x-iR)}\,\mathrm{d}x + \int_{-R}^{R} \frac{\cos(\lambda\pi - i\lambda y)}{\cos(\alpha\pi - i\alpha y)} e^{p(\pi+iy)} i\,\mathrm{d}y$$
$$+ \int_{\pi}^{0} \frac{\cos(\lambda x + i\lambda R)}{\cos(\alpha x + i\alpha R)} e^{p(x+iR)}\,\mathrm{d}x + \int_{R}^{-R} \frac{\cos(0 + i\lambda y)}{\cos(0 + i\alpha y)} e^{p(0+iy)} i\,\mathrm{d}y. \tag{7.30}$$

Consider the two integrals;

$$\int_0^{\pi} \frac{\cos(\lambda x - i\lambda R)}{\cos(\alpha x - i\alpha R)} e^{p(x-iR)}\,\mathrm{d}x, \tag{7.31}$$

and

$$\int_{\pi}^{0} \frac{\cos(\lambda x + i\lambda R)}{\cos(\alpha x + i\alpha R)} e^{p(x+iR)}\,\mathrm{d}x. \tag{7.32}$$

Referring back to equation (7.31):

$$\begin{aligned}
\left|\int_0^{\pi} \frac{\cos(\lambda x - i\lambda R)}{\cos(\alpha x - i\alpha R)} e^{p(x-iR)}\,\mathrm{d}x\right| &= \left|\int_0^{\pi} \left|\frac{e^{i(\lambda x - i\lambda R)} + e^{-i(\lambda x - i\lambda R)}}{e^{i(\alpha x - i\alpha R)} + e^{-i(\alpha x - i\alpha R)}} e^{p(x-iR)}\right| \mathrm{d}x\right| \\
&\le \left|\int_0^{\pi} \frac{\left|e^{i(\lambda x - i\lambda R)}\right| + \left|e^{-i(\lambda x - i\lambda R)}\right|}{\left|\left|e^{i(\alpha x - i\alpha R)}\right| - \left|e^{-i(\alpha x - i\alpha R)}\right|\right|} \left|e^{p(x-iR)}\right| \mathrm{d}x\right| \\
&= \left|\int_0^{\pi} \frac{\left|e^{\lambda R}\right| + \left|e^{-\lambda R}\right|}{\left|\left|e^{\alpha R}\right| - \left|e^{-\alpha R}\right|\right|} \left|e^{px}\right| \mathrm{d}x\right|.
\end{aligned} \tag{7.33}$$

Taking limits as $R \to \infty$, we obtain:

$$\left| \int_0^{\pi} \frac{e^{\lambda R}}{e^{\alpha R}} e^{px} \, \mathrm{d}x \right| \to 0, \tag{7.34}$$

since $\alpha > \lambda$.

Similarly, considering equation (7.32), we have :

$$\begin{aligned}
\left| \int_{\pi}^{0} \frac{\cos(\lambda x + i\lambda R)}{\cos(\alpha x + i\alpha R)} e^{p(x+iR)} \, \mathrm{d}x \right| &= \left| \int_0^{\pi} \left| \frac{e^{i(\lambda x + i\lambda R)} + e^{-i(\lambda x + i\lambda R)}}{e^{i(\alpha x + i\alpha R)} + e^{-i(\alpha x + i\alpha R)}} e^{p(x+iR)} \right| \, \mathrm{d}x \right| \\
&\leq \left| \int_{\pi}^{0} \frac{\left| e^{i(\lambda x + i\lambda R)} \right| + \left| e^{-i(\lambda x + i\lambda R)} \right|}{\left| \left| e^{i(\alpha x + i\alpha R)} \right| - \left| e^{-i(\alpha x + i\alpha R)} \right| \right|} \left| e^{p(x+iR)} \right| \, \mathrm{d}x \right| \\
&= \left| \int_{\pi}^{0} \frac{\left| e^{-\lambda R} \right| + \left| e^{\lambda R} \right|}{\left| \left| e^{-\alpha R} \right| - \left| e^{\alpha R} \right| \right|} \left| e^{px} \right| \, \mathrm{d}x \right|,
\end{aligned} \tag{7.35}$$

which, once again tends to zero as $R \to \infty$.

Combining the above and taking limits as $R \to \infty$ we can write equation (7.30) as:

$$\begin{aligned}
\int_{\Gamma} \frac{\cos(\lambda z)}{\cos(\alpha z)} e^{pz} \, \mathrm{d}z &= \int_{-\infty}^{\infty} \frac{\cos(\lambda \pi - i\lambda y)}{\cos(\alpha \pi - i\alpha y)} e^{p(\pi + iy)} i \, \mathrm{d}y \\
&+ \int_{\infty}^{-\infty} \frac{\cos(i\lambda y)}{\cos(i\alpha y)} e^{ipy} i \, \mathrm{d}y.
\end{aligned} \tag{7.36}$$

If we switch the limits of integration we get:

$$\begin{aligned}\int_{\Gamma} \frac{\cos(\lambda z)}{\cos(\alpha z)} e^{pz}\,\mathrm{d}z &= \int_{-\infty}^{\infty} \frac{\cos(\lambda\pi - i\lambda y)}{\cos(\alpha\pi - i\alpha y)} e^{p(\pi+iy)} i\,\mathrm{d}y \\ &- \int_{-\infty}^{\infty} \frac{\cos(i\lambda y)}{\cos(i\alpha y)} e^{ipy} i\,\mathrm{d}y. \end{aligned} \tag{7.37}$$

We refer to double angle formulae for trigonometric functions to deduce that:

$$\cos(\lambda\pi - i\lambda y) = \cos(\lambda\pi)\cos(i\lambda\ y) + \sin(\lambda\pi)\sin(i\lambda y) \tag{7.38}$$

$$\cos(\alpha\pi - i\lambda y) = \cos(\alpha\pi)\cos(i\alpha\ y) + \sin(\alpha\pi)\sin(i\alpha y)\,. \tag{7.39}$$

Assuming that α is an integer, equations (7.38) and (7.59) become:

$$\cos(\lambda\pi - i\lambda y) = \cos(\lambda\pi)\cos(i\lambda\ y) + \sin(\lambda\pi)\sin(i\lambda y) \tag{7.40}$$

$$\cos(\alpha\pi - i\lambda y) = \cos(\alpha\pi)\cos(i\alpha\ y)\,. \tag{7.41}$$

Substituting these back into equation (7.37);

$$\begin{aligned}\int_{\Gamma} \frac{\cos(\lambda z)}{\cos(\alpha z)} e^{pz}\,\mathrm{d}z &= \int_{-\infty}^{\infty} \frac{\cos(\lambda\pi)\cos(i\lambda y) + \sin(\lambda\pi)\sin(i\lambda y)}{\cos(\alpha\pi)\cos(i\alpha y)} e^{p(\pi+iy)} \\ &- \frac{\cos(i\lambda y)}{\cos(i\alpha y)} e^{ipy} i\,\mathrm{d}y. \end{aligned} \tag{7.42}$$

After rearrangement, we obtain;

$$\int_\Gamma \frac{\cos(\lambda z)}{\cos(\alpha z)} e^{pz}\,\mathrm{d}z = [Ae^{p\pi} - 1]\, i \int_{-\infty}^{\infty} \frac{\cos(i\lambda y)}{\cos(i\alpha y)} e^{ipy}\,\mathrm{d}y + B\int_{-\infty}^{\infty} \frac{\sin(i\lambda y)}{\cos(i\alpha y)} e^{p(\pi+iy)} i\,\mathrm{d}y, \tag{7.43}$$

where

$$A = \frac{\cos(\lambda\pi)}{\cos(\alpha\pi)} \text{ and } B = \frac{\sin(\lambda\pi)}{\cos(\alpha\pi)}.$$

Using the expression we obtained above in equation (7.29) we can express the above equation as:

$$\begin{aligned}\int_\Gamma \frac{\cos(\lambda z)}{\cos(\alpha z)} e^{pz}\,\mathrm{d}z &= [Ae^{p\pi} - 1]\, i \int_{-\infty}^{\infty} \frac{\cos(i\lambda y)}{\cos(i\alpha y)} e^{ipy}\,\mathrm{d}y + B\int_{-\infty}^{\infty} \frac{\sin(i\lambda y)}{\cos(i\alpha y)} e^{p(\pi+iy)} i\,\mathrm{d}y \\ &= -\frac{2\pi i}{\alpha} e^{\pi p/2\alpha} \cos\left(\frac{\pi\lambda}{2\alpha}\right). \end{aligned} \tag{7.44}$$

Rearranging, we obtain:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos(i\lambda y)}{\cos(i\alpha y)} e^{ipy}\,\mathrm{d}y &= \frac{2\pi i}{\alpha\,[1 - Ae^{p\pi}]\, i} e^{\pi p/2\alpha} \cos\left(\frac{\pi\lambda}{2\alpha}\right) \\ &+ \frac{B}{[1 - Ae^{p\pi}]} \int_{-\infty}^{\infty} \frac{\sin(i\lambda y)}{\cos(i\alpha y)} e^{p(\pi+iy)}\,\mathrm{d}y, \end{aligned} \tag{7.45}$$

and noting that

$$\begin{aligned}
\mathscr{F}^{-1}\left[\operatorname{sech}(\alpha y)\cosh(\lambda y)\right] &= \int_{-\infty}^{\infty} \operatorname{sech}(\alpha y)\cosh(\lambda y)\, e^{ipy}\, \mathrm{d}y \\
&= \int_{-\infty}^{\infty} \frac{\cos(i\lambda y)}{\cos(i\alpha y)} e^{ipy}\, \mathrm{d}y, \qquad (7.46)
\end{aligned}$$

we can deduce that:

$$\begin{aligned}
\mathscr{F}^{-1}\left[\operatorname{sech}(\alpha y)\cosh(\lambda y)\right] &= \frac{2\pi}{\alpha\left[1 - Ae^{p\pi}\right]} e^{\pi p/2\alpha} \cos\left(\frac{\pi\lambda}{2\alpha}\right) \\
&+ \frac{Be^{p\pi}}{\left[1 - Ae^{p\pi}\right]} \int_{-\infty}^{\infty} \frac{\sin(i\lambda y)}{\cos(i\alpha y)} e^{piy}\, \mathrm{d}y. \qquad (7.47)
\end{aligned}$$

Finally, we calculate the integral in equation (7.47):

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin(i\lambda y)}{\cos(i\alpha y)} e^{piy}\, \mathrm{d}y &= \int_{-\infty}^{\infty} \frac{i\sinh(\lambda y)}{\cosh(\alpha y)} e^{piy}\, \mathrm{d}y = i\int_{-\infty}^{\infty} \frac{e^{\lambda y} - e^{-\lambda y}}{e^{\alpha y} + e^{-\alpha y}} e^{piy}\, \mathrm{d}y \\
&= i\int_{-\infty}^{\infty} \frac{e^{\lambda y} e^{piy}}{e^{\alpha y} + e^{-\alpha y}}\, \mathrm{d}y - i\int_{-\infty}^{\infty} \frac{e^{-\lambda y} e^{piy}}{e^{\alpha y} + e^{-\alpha y}}\, \mathrm{d}y \\
&= i\int_{-\infty}^{\infty} \frac{e^{-iy(i\lambda - p)}}{e^{\alpha y} + e^{-\alpha y}}\, \mathrm{d}y - i\int_{-\infty}^{\infty} \frac{e^{-iy(-p - i\lambda)}}{e^{\alpha y} + e^{-\alpha y}}\, \mathrm{d}y \\
&= \frac{i}{2}\int_{-\infty}^{\infty} \operatorname{sech}(\alpha y)\, e^{-icy}\, \mathrm{d}y - \frac{i}{2}\int_{-\infty}^{\infty} \operatorname{sech}(\alpha y)\, e^{-idy}\, \mathrm{d}y \\
&= \frac{i}{2}\mathscr{F}\left[\operatorname{sech}(\alpha y)\right](c) - \frac{i}{2}\mathscr{F}\left[\operatorname{sech}(\alpha y)\right](d), \qquad (7.48)
\end{aligned}$$

where $c = i\lambda - p$ and $d = -p - i\lambda$.

Now, we have already seen from the first calculation of the appendix, Fourier Transform 1, that the Fourier transform of sech is another sech function, hence, we can deduce that:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin(i\lambda y)}{\cos(i\alpha y)} e^{piy}\,\mathrm{d}y &= \frac{i}{2}\mathscr{F}\left[\operatorname{sech}(\alpha y)\right](c) - \frac{i}{2}\mathscr{F}\left[\operatorname{sech}(\alpha y)\right](d) \\
&= \frac{i}{2}\alpha\pi\operatorname{sech}\left(\frac{\pi c}{2}\right) - \frac{i}{2}\alpha\pi\operatorname{sech}\left(\frac{\pi d}{2}\right) \\
&= \frac{i}{2}\alpha\pi\operatorname{sech}\left(\frac{\pi(i\lambda - p)}{2}\right) - \frac{i}{2}\alpha\pi\operatorname{sech}\left(\frac{\pi(-p - i\lambda)}{2}\right) \\
&= \frac{i\pi\alpha}{2}\left[\operatorname{sech}\left(\frac{i\pi\lambda - \pi p}{2}\right) - \operatorname{sech}\left(\frac{-p\pi - i\pi\lambda}{2}\right)\right]. \qquad (7.49)
\end{aligned}$$

We evaluate the sum of sech functions above:

$$\begin{aligned}
\operatorname{sech}\left(\frac{i\pi\lambda - \pi p}{2}\right) - \operatorname{sech}\left(\frac{-p\pi - i\pi\lambda}{2}\right) &= \frac{2}{e^{i\pi\lambda/2}e^{-p\pi/2} + e^{-i\pi\lambda/2}e^{p\pi/2}} \\
&- \frac{2}{e^{-i\pi\lambda/2}e^{-p\pi/2} + e^{i\pi\lambda/2}e^{p\pi/2}}. \qquad (7.50)
\end{aligned}$$

Bringing terms together over a common denominator and after some manipulation, we can express:

$$\operatorname{sech}\left(\frac{i\pi\lambda - \pi p}{2}\right) - \operatorname{sech}\left(\frac{-p\pi - i\pi\lambda}{2}\right) = \frac{4i\sin(\pi\lambda/2)\sinh(p\pi/2)}{\cos(\lambda\pi) + \cosh(p\pi)}. \qquad (7.51)$$

Substituting equation (7.51) back into equation (7.49), we obtain:

$$\int_{-\infty}^{\infty} \frac{\sin(i\lambda y)}{\cos(i\alpha y)} e^{piy}\, \mathrm{d}y = \frac{i\pi\alpha}{2}\left[\frac{4i\sin(\pi\lambda/2)\sinh(p\pi/2)}{\cos(\lambda\pi)+\cosh(p\pi)}\right]$$
$$= -2\pi\alpha\left[\frac{\sin(\pi\lambda/2)\sinh(p\pi/2)}{\cos(\lambda\pi)+\cosh(p\pi)}\right]. \tag{7.52}$$

Substituting equation (7.52) into (7.47), we have demonstrated that:

$$\mathscr{F}^{-1}\left[\operatorname{sech}(\alpha y)\cosh(\lambda y)\right] = -2\frac{\pi\alpha B e^{p\pi}}{[1-Ae^{p\pi}]}\left[\frac{\sin(\pi\lambda/2)\sinh(p\pi/2)}{\cos(\lambda\pi)+\cosh(p\pi)}\right]$$
$$+ \frac{2\pi}{\alpha[1-Ae^{p\pi}]} e^{\pi p/2\alpha}\cos\left(\frac{\pi\lambda}{2\alpha}\right). \tag{7.53}$$

7.4 Proof of Lemma 3.11

We refer the reader to Lemma 3.11 that we made use of in Chapter 3:

Lemma 3.12. *For values of θ such that $\left|\cosh\left(\lambda r e^{i\theta}\right)\right| < 1$, the following integral is finite:*

$$\begin{aligned} & -\int_0^{2\pi}\Big|_{\left\{\theta:\left|\cosh\left(\lambda r e^{i\theta}\right)\right|<1\right\}} \ln\left|\cosh\left(\lambda r e^{i\theta}\right)\right| \mathrm{d}\theta \\ \leq & -\int_0^{2\pi}\Big|_{\left\{\theta:\left|\cosh\left(\lambda e^{i\theta}\right)\right|<1\right\}} \ln\left|\cosh\left(\lambda e^{i\theta}\right)\right| \mathrm{d}\theta \\ \equiv & \ \tilde{W} < \infty. \end{aligned} \tag{7.54}$$

Proof of Lemma 3.12

We begin by making estimates on $\left|\cosh\left(\lambda r e^{i\theta}\right)\right|$. We note that we can express $\left|\cosh\left(\lambda r e^{i\theta}\right)\right|$ as:

$$\begin{aligned}
\left|\cosh\left(\lambda re^{i\theta}\right)\right| &= \sqrt{\cos^2\left(\lambda r\sin\left(\theta\right)\right)\cosh^2\left(\lambda r\cos\left(\theta\right)\right)+\sin^2\left(\lambda r\sin\left(\theta\right)\right)\sinh^2\left(\lambda r\cos\left(\theta\right)\right)} \\
&= \sqrt{\left[1-\sin^2\left(\lambda r\sin\left(\theta\right)\right)\right]\cosh^2\left(\lambda r\cos\left(\theta\right)\right)+\sin^2\left(\lambda r\sin\left(\theta\right)\right)\sinh^2\left(\lambda r\cos\left(\theta\right)\right)} \\
&= \sqrt{\cosh^2\left(\lambda r\cos\left(\theta\right)\right)-\sin^2\left(\lambda r\sin\left(\theta\right)\right)\left[\cosh^2\left(\lambda r\cos\left(\theta\right)\right)-\sinh^2\left(\lambda r\cos\left(\theta\right)\right)\right]} \\
&= \sqrt{\cosh^2\left(\lambda r\cos\left(\theta\right)\right)-\sin^2\left(\lambda r\sin\left(\theta\right)\right)}. \qquad (7.55)
\end{aligned}$$

Since we are considering values of r and θ such that $\left|\cosh\left(\lambda re^{i\theta}\right)\right| < 1$, then we can consider values of r and θ such that $\sqrt{\cosh^2\left(\lambda r\cos\left(\theta\right)\right)-\sin^2\left(\lambda r\sin\left(\theta\right)\right)} < 1$.

We can demonstrate graphically (see figure 7.3) that for $0.5\sqrt{\cosh^2\left(\lambda\cos\left(\theta\right)\right)-\sin^2\left(\lambda\sin\left(\theta\right)\right)} \leq 1$ (blue curve), it is true that:

$$\cosh^2\left(\lambda r\cos\left(\theta\right)\right)-\frac{1}{4}\cosh^2\left(\lambda\cos\left(\theta\right)\right)+\frac{1}{4}\sin^2\left(\lambda\sin\left(\theta\right)\right)-\sin^2\left(\lambda r\sin\left(\theta\right)\right)\geq 0,$$

which is represented by the red curve in figure 7.3, which implies that:

$$\cosh^2\left(\lambda r\cos\left(\theta\right)\right)-\sin^2\left(\lambda r\sin\left(\theta\right)\right)\geq\frac{1}{4}\left[\cosh^2\left(\lambda\cos\left(\theta\right)\right)-\sin^2\left(\lambda\sin\left(\theta\right)\right)\right].$$

We deduce from this that:

$$\sqrt{\cosh^2\left(\lambda r\cos\left(\theta\right)\right)-\sin^2\left(\lambda r\sin\left(\theta\right)\right)}\geq\frac{1}{2}\sqrt{\cosh^2\left(\lambda\cos\left(\theta\right)\right)-\sin^2\left(\lambda\sin\left(\theta\right)\right)}.$$

Substituting this back into equation (7.55), we obtain:

$$
\begin{aligned}
\left|\cosh\left(\lambda r e^{i\theta}\right)\right| &\geq \frac{1}{2}\sqrt{\cosh^2\left(\lambda\cos\left(\theta\right)\right)-\sin^2\left(\lambda\sin\left(\theta\right)\right)}\\
&\geq \frac{1}{2}\sqrt{\cosh^2\left(\lambda\cos\left(\theta\right)\right)-1}\\
&= \frac{1}{2}\sqrt{\sinh^2\left(\lambda\cos\left(\theta\right)\right)}\\
&= \frac{1}{2}\left|\sinh\left(\lambda\cos\left(\theta\right)\right)\right|. \qquad (7.56)
\end{aligned}
$$

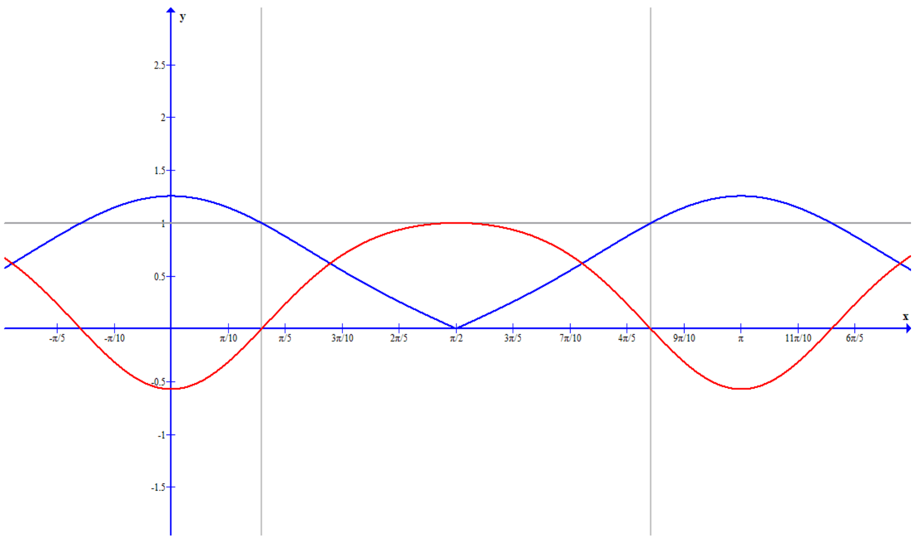

Figure 7.3: Estimates for Lemma 3.12

Figure 7.3 demonstrates that for $0.5\sqrt{\cosh^2\left(\lambda\cos\left(\theta\right)\right)-\sin^2\left(\lambda\sin\left(\theta\right)\right)} \leq 1$ (blue curve), that: $\cosh^2\left(\lambda r\cos\left(\theta\right)\right)-\frac{1}{4}\cosh^2\left(\lambda\cos\left(\theta\right)\right)+\frac{1}{4}\sin^2\left(\lambda\sin\left(\theta\right)\right)-\sin^2\left(\lambda r\sin\left(\theta\right)\right) \geq 0$ (red curve) regardless of what value we chose for r.

If follows that:

$$
\begin{aligned}
-\int_0^{2\pi}\Big|_{\theta:\left|\cosh\left(\lambda re^{i\theta}\right)\right|<1} \ln\left|\cosh\left(\lambda re^{i\theta}\right)\right| \,\mathrm{d}\theta \;&\leq\; -\int_0^{2\pi}\Big|_{\theta:|V(\theta)|<2} \ln\left|\frac{1}{2}\sinh\left(\lambda\cos\left(\theta\right)\right)\right| \,\mathrm{d}\theta \\
&=\; -\int_0^{2\pi}\Big|_{\theta:|V(\theta)|<1} \ln\left|\sinh\left(\lambda\cos\left(\theta\right)\right)\right| \,\mathrm{d}\theta \\
&-\; 2\pi\ln\left(1/2\right), \qquad (7.57)
\end{aligned}
$$

where $V(\theta) = \sinh(\lambda\cos(\theta))$.

An important thing to note here is that the integral of $\ln|\sinh(\lambda\cos(\theta))|$ is independent of r, and so will only contribute a constant to our $\rho(r)$, (providing the integral is finite).

We demonstrate that:

$$
\int_0^{2\pi}\Big|_{\theta:|V(\theta)|<1} \ln\left|\sinh\left(\lambda\cos\left(\theta\right)\right)\right| \,\mathrm{d}\theta \qquad (7.58)
$$

is finite.

We introduce the following function:

$$
P_1(\theta) = \begin{cases} -1/2\left(\theta+\pi/2\right)\left(\theta-\pi/2\right), & \text{if } 0<\theta\leq\pi/2 \\ -1/2\left(\theta-3\pi/2\right)\left(\theta-\pi/2\right), & \text{if } \pi/2<\theta<3\pi/2 \\ -1/2\left(\theta-3\pi/2\right)\left(\theta-5\pi/2\right), & \text{if } 3\pi/2\leq\theta<2\pi. \end{cases}
$$

We note that $|P_1(\theta)| \leq |\sinh(\lambda\cos(\theta))|$ for all θ, and hence, we can deduce that:

$$
\begin{aligned}
I_P &= -\int_0^{2\pi}\bigg|_{\theta:|V(\theta)|<1} \ln\left|\sinh\left(\lambda\cos\left(\theta\right)\right)\right|\,\mathrm{d}\theta \\
&\leq -\int_0^{2\pi}\bigg|_{\theta:|P_1(\theta)|<1} \ln\left|P_1\left(\theta\right)\right|\,\mathrm{d}\theta \\
&= -\int_0^{\pi/2}\bigg|_{\theta:|P_1(\theta)|<1} \ln\left|P_1\left(\theta\right)\right|\,\mathrm{d}\theta - \int_{\pi/2}^{3\pi/2}\bigg|_{\theta:|P_1(\theta)|<1} \ln\left|P_1\left(\theta\right)\right|\,\mathrm{d}\theta \\
&- \int_{3\pi/2}^{2\pi}\bigg|_{\theta:|P_1(\theta)|<1} \ln\left|P_1\left(\theta\right)\right|\,\mathrm{d}\theta \\
&= -\tilde{W} < \infty. \qquad (7.59)
\end{aligned}
$$

Since it is obvious that $\ln\left|P_1\left(\theta\right)\right|$ is integrable, I_P is simply some constant (negative), which we call $\tilde{W}$ (positive).

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